

MINISTRY OF EDUCATION AND TRAINING
UNIVERSITY OF TRANSPORT AND COMMUNICATIONS

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**DEVELOPMENT OF THE STOCHASTIC
FINITE ELEMENT METHOD FOR
MECHANICAL BEHAVIOR ANALYSIS OF
BEAMS AND COLUMNS CONSIDERING
SPATIALLY RANDOM MATERIAL
PROPERTIES**

Major : Engineering Mechanics

Code No : 9520101

DOCTORAL THESIS SUMMARY

HA NOI - 2026

This research was completed at the University of Transport and Communications

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The thesis was defended before the Doctoral-Level Evaluation Council at the
University of Transport and Communications at

The thesis can be referred at:

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INTRODUCTION

1. Research Motivation

Traditional deterministic models neglect the spatial variability of material properties, failing to capture the inherent uncertainty in structural behavior. Although the Stochastic Finite Element Method (SFEM) and random field discretization techniques have been well-developed, current analyses of beam and column members remain largely confined to one-dimensional (1D) axial random fields. To accurately simulate the spatial randomness of material properties, this dissertation develops an SFEM framework that directly integrates multi-dimensional (2D, 3D) random fields. This contribution not only addresses the limitations of existing models but also provides a reliable predictive tool for structural design.

2. Research Objectives

The objective of this dissertation is to develop the Stochastic Finite Element Method (SFEM) for analyzing the stochastic mechanical responses of beams and columns, accounting for the multi-dimensional (2D, 3D) spatial variability of material properties. Specifically, the research employs the weighted integral method to discretize continuous random fields, combined with a first-order Taylor expansion perturbation method, to directly formulate the statistical characteristics of the structures, including displacements, free vibration eigenvalues, and critical buckling loads.

To validate the reliability of the proposed method, the analytical results are compared with published literature and cross-verified via Monte Carlo Simulations (MCS) coupled with an independently developed Spectral Representation Method (SRM). Subsequently, a parametric analysis is conducted to quantitatively assess the impact of random field parameters (including the coefficient of variation, correlation length, and cross-correlation) on the stochastic behavior of beams and columns.

3. Research Object and Scope

The subjects of this dissertation are planar structural members (beams and columns) with constant rectangular cross-sections, whose spatially varying material properties (elastic modulus and mass density) are modeled as multi-dimensional, stationary, cross-correlated Gaussian random fields. Based on this model, the dissertation addresses three fundamental problems: (i) stochastic static responses, (ii) linear dynamic characteristics, and (iii) Euler buckling of axially loaded columns.

Within the research scope, the SFEM framework is developed using Euler–Bernoulli beam theory for planar bending, neglecting secondary effects such as torsion and eccentricity between the geometric centroid and the shear/compression center. Additionally, the coefficient of variation (COV) for the material random fields is limited to $\text{COV} \leq 20\%$. The study focuses on determining the statistical moments (mean, variance, and COV) of displacements, free vibration eigenvalues, and critical buckling loads to evaluate the impact of spatial material variability on the stochastic behavior of these members.

4. Scientific Contributions and Practical Implications of the Dissertation

This dissertation contributes novel insights to stochastic mechanics by developing an SFEM framework to analyze the stochastic responses of beams and columns under spatially varying material properties (2D, 3D). Thereby, the research helps refine the theoretical framework for structural analysis—encompassing static, linear dynamic, and stability problems—under conditions of uncertainty.

Practically, the dissertation provides a robust foundation for simulating and predicting the stochastic responses of beams and columns, supporting reliability-based structural design. Modeling with spatial random fields accurately captures the material heterogeneity inherent in manufacturing and construction processes. The proposed SFEM framework can be directly applied to the design of safety-critical structures, as well as facilitating proactive structural monitoring and maintenance.

5. Research Methodology

To fulfill the defined scientific objectives, this dissertation employs an analytical mechanics framework based on the Euler-Bernoulli beam theory to establish the governing differential equations. Subsequently, the weighted integral technique is developed to discretize the multi-dimensional (2D, 3D) spatial random fields of the material properties into fundamental random variables. This establishes the foundation for formulating the Stochastic Finite Element Method (SFEM) in conjunction with the first-order perturbation approach, facilitating the direct computation of the statistical characteristics of the mechanical responses without requiring iterative procedures. Finally, the accuracy and convergence of the proposed SFEM model are independently validated against Monte Carlo Simulation (MCS) and published studies, serving as a prerequisite for quantitative parametric investigations aimed at establishing the laws governing the influence of random fields on structural behavior.

CHAPTER I: OVERVIEW OF STOCHASTIC STRUCTURAL ANALYSIS

1.1. Random Variables

A random experiment is a trial performed under identical conditions whose outcome cannot be predicted with certainty. The probability space of a random quantity is denoted by (Ω, F, P) where Ω is the sample space, F is the sigma-algebra, and P is the probability measure mapping into the interval $[0, 1]$. A random variable X (continuous or discrete) is a function defined on Ω . For a continuous random variable X , its fundamental characteristics include the probability density function (PDF), $f_X(x)$, and the cumulative distribution function (CDF), $F_X(x)$.

1.2. Random Fields

A random function is classified as a stochastic process (if the independent variable is time, t) or a random field, $T(\mathbf{x}, \theta)$, if the independent variable is a spatial coordinate, $\mathbf{x}$. Statistically, the properties of a random field are characterized by its mean function, variance function, and spatial correlation descriptors (autocorrelation, autocovariance, and cross-correlation). A Gaussian random field is completely defined by these characteristics; a stationary random field requires its mean and variance to be constant, and its autocorrelation function to depend solely on the spatial separation distance.

1.3. Overview of Random Field Representation Methods

1.3.1. Representation Methods for Gaussian Random Fields

1.3.1.1. Point Representation Methods

The random field $T(\mathbf{x}, \theta)$ is discretized into random variables at one or several points $\mathbf{x}_i$ within an element. The magnitude, mean, and variance of the random variable are determined directly from the random field values at $\mathbf{x} = \mathbf{x}_i$. The spatial correlation between two points is approximated via the autocorrelation function.

1.3.1.2. Averaging Representation Methods

a. Spatial Averaging Method

The spatial averaging method [121] approximates the random field $T(\mathbf{x}, \theta)$ as a single random variable X_e by averaging it via integration over the element domain V_e . Based on a mesh comprising N_e , the random field is represented by a finite random vector $\{X_1, X_2, \dots, X_{N_e}\}$. The mean vector and covariance matrix of this vector are determined by integrating the mean and covariance functions of the original random field over the element domain.

b. Weighted Integral Method

The weighted integral method converts a continuous random field into a finite set of basic random variables X_{ϕ} . This technique integrates the random field weighted by spatial polynomials over the element domain V_e . Consequently, the element stiffness matrix is decomposed, wherein the stochastic component is expressed as a linear combination of these basic random variables. This approach completely decouples the stochastic and deterministic parts, enabling efficient integration into the SFEM framework for analyzing structures with multi-dimensional random materials while controlling the computational effort based on the element mesh.

1.3.1.3. Series Expansion Methods for Random Field Representation

a. Karhunen–Loève (K-L) Expansion Method

The Karhunen–Loève (K-L) expansion method represents a continuous random field through a finite set of uncorrelated random variables, based on the spectral analysis of the autocovariance function via the Fredholm integral equation. Due to the orthogonality of the eigenfunctions, the random field is decoupled into deterministic and stochastic components. In SFEM, truncating the K-L series at the first N terms yields an optimal approximation in the least-squares sense with minimum error, although the inevitable consequence is that the approximated variance is always strictly less than that of the original field.

b. Orthogonal Series Expansion Method

The orthogonal series expansion method approximates a random field based on a predefined set of orthogonal basis functions. To eliminate the correlation among the random variables, spectral decomposition of the autocovariance matrix is applied. Consequently, the random field is compactly represented by a truncated series of N terms, which is a linear combination of basis functions, spectral characteristics, and a set of completely independent standard normal random variables.

c. Extended Optimal Linear Estimation Method

The optimal linear estimation method discretizes the random field into a random vector at the nodes of an approximation mesh. By applying spectral decomposition to the autocovariance matrix, this nodal vector is decomposed via its eigenvalues, eigenvectors, and a set of independent standard normal random variables. Finally, the random field at any arbitrary point $\mathbf{x}$ is linearly approximated based on the spectral properties of the basis nodes and the covariance vector correlating point $\mathbf{x}$ with those nodes.

d. Spectral Representation Method (SRM)

The spectral representation method approximates a stationary random field using a finite trigonometric series, where the amplitudes are derived from the power spectral density (PSD) function, and the uncertainty is introduced solely through uniformly distributed random phase angles. The computational process is optimized by truncating the frequency domain at an upper cut-off frequency ω_u , ensuring the generated samples strictly preserve the original correlation structure.

1.3.2. Representation Methods for Non-Gaussian Random Fields

1.3.2.1. Translation Methods

a. Correlation Transformation Method (Translation Method)

The translation method represents a stationary non-Gaussian random field from a standard Gaussian field via a pointwise iso-probabilistic transformation. Mathematically, the value of the non-Gaussian field at a specific location is directly mapped from the Gaussian field through the standard normal cumulative distribution function (CDF) and the inverse of the target marginal CDF. The most rigorous constraint of this technique is that the power spectral

density (PSD) and the marginal distribution of the non-Gaussian field must satisfy a statistical correlation compatibility condition, which is analytically established via the integration of the bivariate normal PDF.

b. Iterative Translation Method

This technique simulates non-Gaussian random fields using an iterative mapping procedure that integrates the Karhunen-Loève expansion into the iterative loop, aiming to overcome PDF distortion in fields with high skewness. This improvement enables the algorithm to successfully generate sample realizations that accurately and simultaneously match both target criteria: the marginal distribution and the PSD.

1.3.2.2. Polynomial Chaos Expansion (PCE) Methods

The polynomial chaos expansion (PCE) method solves stochastic differential equations by approximating the non-Gaussian field via the Askey scheme of orthogonal polynomials. By applying Galerkin projection, the complex stochastic system is entirely converted into a set of deterministic equations, optimizing the computational dimensionality and achieving a superior exponential convergence rate.

1.4. Overview of the Stochastic Finite Element Method (SFEM)

1.4.1. Monte Carlo Simulation (MCS) Method

The MCS method evaluates structural responses by generating a large set of realizations from the input random field. For each sample θ , the global structural equilibrium equation, $[K(\theta)]\{u(\theta)\}=\{P\}$, is formulated and solved to obtain the displacement vector. Through statistical processing of this output dataset, the stochastic characteristics of the structure, such as the mean, variance, and COV, are determined. Despite its universality and high reliability, the primary barrier of this technique is the immense computational burden for high-dimensional systems.

1.4.2. Perturbation Method

The perturbation method estimates structural responses by expanding the quantities within the global equilibrium equation via a Taylor series around their mean values and equating terms of the same order. This technique converts the stochastic system into a sequence of deterministic equations, allowing for the direct computation of the mean and covariance matrix of displacements through approximated series.

1.4.3. Spectral Stochastic Finite Element Method (SSFEM)

SSFEM combines the K - L expansion and Polynomial Chaos Expansion (PCE), employing Galerkin projection to transform the stochastic system into a large-scale deterministic algebraic problem. This method allows for the direct determination of the PDF with an exponential convergence rate, vastly outperforming iterative simulation techniques. However, the combinatorial explosion in computational cost as the dimensionality or polynomial degree increases remains a major challenge, necessitating sparse solver algorithms to optimize performance.

1.5. Overview of Literature Related to the Dissertation

1.5.1. Overview of SFEM Applications in Scientific Fields

The Stochastic Finite Element Method (SFEM) has been widely applied to quantify uncertainty across various disciplines. In materials science, this method evaluates the impact of random defects on the reliability of composite materials, porous materials, and metal foams [60, 90, 110]. In biomechanics, SFEM supports the quantification of property dispersion in bones, spines, and implants [74, 100, 38], as well as predicting the risk of aortic rupture. In civil and geotechnical engineering, SFEM addresses problems such as rock foundation settlement [54], groundwater flow simulation [130], critical load determination for thin-walled structures [28, 122], and fatigue life assessment of steel bridge systems [57].

1.5.2. Global Research Trends

1.5.2.1. Overview of Stochastic Static Analysis of Beam Structures

Uncertainty analysis of beam material properties has transitioned from simple random variable assumptions to random field models. Initially, the local averaging technique [121] exhibited limitations due to its dependency on the element mesh size. This was overcome by the weighted integral approach [32, 35, 120], which expresses the equivalent stiffness matrix and load vector via integrated random variables. This approach yields mathematically rigorous solutions independent of both meshing and the correlation length of the random field. Concurrently, exact analytical solutions have been extensively developed to establish verification benchmarks and eliminate discretization errors. Based on deterministic foundations [44], closed-form formulations for stochastic stiffness [16] and flexibility matrices [40] have been established. Particularly for materials with a high coefficient of variation (COV), variational principle-based methods [99] allow for the direct determination of the mean and covariance functions of displacements with superior accuracy compared to traditional perturbation methods. Currently, the research focus is shifting toward multi-dimensional spatial problems and experimental integration. Local averaging has been extended to 3D random fields using Neumann expansion combined with Monte Carlo Simulation (MCS) [25], alongside assessments of how spatial variability impacts structural reliability [105]. Modern research also emphasizes updating statistical models via experimental data, damage identification [126, 127], and applying the maximum entropy principle coupled with MCS to accurately determine the probability distribution of mechanical responses under stochastic loads [30].

1.5.2.2. Overview of Stochastic Vibration of Beams and Stochastic Buckling of Axially Loaded Columns

In stochastic dynamics and stability, the perturbation method combined with SFEM has been widely applied to complex members such as non-uniform or rotating beams [26, 63]. Approaches based on the dynamic stiffness matrix and Fourier transform provide exact solutions in the frequency domain [15, 58], while integrating the K-L expansion into a double SSFEM framework [14, 91] achieved a computational breakthrough. For stability problems, foundational studies [18, 136] determined the critical buckling loads of columns considering uncertain stiffness and geometric imperfections, which were later extended to members with random initial curvatures [37, 69]. Notably, the application of the weighted integral method [116] fundamentally resolved the mesh-dependency inherent in traditional point discretization techniques for static stability problems.

Recently, research trends have expanded into interacting systems (e.g., beams on elastic foundations, vehicle-bridge interactions) and advanced materials (composites) [82, 117], preliminarily accounting for the spatial variability of material properties. Integrating Isogeometric Analysis (IGA) or SFEM for non-Gaussian fields [45, 62, 103] significantly improved accuracy over classical FEM models. Simultaneously, the rapid rise of Artificial Intelligence (AI) and advanced sampling techniques (e.g., Latin Hypercube) [66, 79, 134] has opened new avenues for optimizing computational costs. However, despite their predictive speed, Hybrid AI models essentially remain "black boxes" or rely solely on simplified beam elements. Consequently, these methods cannot explicitly represent the underlying intrinsic mechanical interactions, particularly when the problem demands the simulation of full three-dimensional (3D) spatial random fields.

1.5.2.3. Overview of Stochastic Analysis of Functionally Graded Material (FGM) Beams

Functionally Graded Material (FGM) beams are characterized by mechanical properties (e.g., elastic modulus and mass density) that vary continuously along the cross-sectional height, typically following exponential or power-law rules [73]. Although traditional studies established a solid foundation for the bending, vibration, and buckling of FGM beams [31, 94], the majority of these models rely on deterministic distribution laws, failing to capture the random variability arising during the manufacturing process.

Representative studies by Noh [88] and the research group of N.V. Thuan and T.D. Hien [84, 85] pioneered the application of SFEM and perturbation techniques to analyze the free vibration of FGM beams under a 1D random elastic modulus field. Beyond probabilistic approaches, research has expanded into fuzzy uncertainty and semi-analytical methods to address scenarios lacking statistical data [95, 137]. Comparisons between deterministic and stochastic analyses on complex FGM members, such as curved beams with porosities [17], demonstrate that neglecting uncertainty can lead to misleading structural safety evaluations. This affirms the critical importance of integrating stochastic factors into the design process to ensure the reliability of systems utilizing modern advanced materials.

1.5.3. Domestic Research Trends

In Vietnam, structural research is strongly shifting from deterministic models toward probabilistic approaches. The MCS method plays a dominant role in analyzing nonlinear dynamic responses, assessing the safety of offshore and tall structures, and investigating the critical loads of members with random elastic connections [5, 6, 8, 13]. To mitigate the massive computational burden associated with MCS, approximate approaches [1] or fuzzy set theory [2] have been flexibly applied to model material and load uncertainties. Additionally, domestic studies delve into sensitivity analyses for structures with defects (e.g., cracks, local stiffness degradation, strut uncertainty) [3, 4, 9], and integrate AI to optimize Tuned Mass Dampers (TMD) under random excitations [10, 11]. Specifically in the transportation sector, vehicle-bridge dynamic interaction problems have made significant progress by addressing road surface roughness and velocity uncertainty via the Probability Density Evolution Method (PDEM) [7, 12]. However, an overarching limitation is that current domestic research predominantly relies on resource-intensive MCS, while material randomness is primarily simplified to random variables or one-dimensional (1D) random fields. The development of a comprehensive SFEM framework that directly solves mechanical problems considering multi-dimensional (2D, 3D) spatial random fields remains a significant, largely unexplored academic gap in Vietnam.

1.5.4. Overall Assessment and Research Problem

A review of the literature reveals that efforts to develop SFEM for beam and column members currently rest on three main pillars: (1) Advancing computational theory by integrating SFEM with higher-order shear deformation theories or isogeometric analysis; (2) Expanding the analytical scope to practical interacting systems, such as stochastic elastic foundations or vehicle-bridge interactions; and (3) Refining models of the spatial random variability of material properties. While the first two pillars have achieved notable progress, the third pillar still contains a distinct academic gap. The vast majority of current studies are confined to 1D random field assumptions, which fail to accurately reflect the complex spatial uncertainty of actual materials. To bridge this gap, this dissertation develops a comprehensive SFEM framework that directly integrates multi-dimensional (2D, 3D) spatial random fields. This contribution not only strengthens the theoretical foundation of stochastic mechanics but also provides a robust predictive tool, meeting the rigorous demands for optimal and reliability-based structural design.

1.6. Conclusion of Chapter I

Methodologically, this dissertation employs stationary Gaussian random fields to model the randomness of material properties, combining the weighted integral method and the perturbation technique to solve the uncertainty

propagation problem, with Monte Carlo Simulation (MCS) serving as the verification benchmark. This approach directly fills the gap left by current 1D studies by integrating the random variability of material properties within a multi-dimensional space, thereby accurately capturing the spatial averaging effect and the intrinsic heterogeneity of actual materials. The dissertation focuses on developing an SFEM theoretical framework based on the weighted integral method to comprehensively address three fundamental problems of beams and columns, accounting for multi-dimensional random distributions:

- ✓ **(i) Stochastic static analysis:** Determining the statistical properties of strains and displacements.
- ✓ **(ii) Stochastic linear dynamic analysis:** Evaluating the dispersion of eigenvalues (free vibration frequencies).
- ✓ **(iii) Stochastic Euler buckling analysis of axially loaded columns:** Determining the statistical properties of the critical Euler buckling load.

Finally, a parametric study is conducted to quantitatively evaluate the impact of the correlation length, COV, and cross-correlation between material fields. The expected outcomes will elucidate the governing role of multi-dimensional spatial randomness, providing a robust scientific foundation for the reliability-based design and assessment of modern structures under uncertainty.

CHAPTER II: DEVELOPMENT OF THE STOCHASTIC FINITE ELEMENT METHOD FOR STATIC AND LINEAR DYNAMIC ANALYSIS OF PLANAR BENDING BEAMS WITH SPATIALLY RANDOM MATERIAL PROPERTIES

2.1. Problem Formulation

Consider a straight Euler-Bernoulli beam with a span length L , a rectangular cross-section of width b and height h , elastic modulus E , and mass density ρ . In the case where $E = E_0$ and $\rho = \rho_0$ are deterministic, the general governing differential equation of motion is given by Eq. (2.1):

$$E_0 I \frac{\partial^4 w(x,t)}{\partial x^4} + \rho_0 A \frac{\partial^2 w(x,t)}{\partial t^2} = q(x,t) \quad (2.1)$$

Due to manufacturing processes, inherent microstructural variations, or aging effects, material properties exhibit spatial randomness. Consequently, the elastic modulus and mass density of the beam are assumed to be 2D random fields, such as (2.2) and (2.3), or three-dimensional (3D) random fields, such as (2.4) and (2.5).

$$E(\mathbf{x}, \theta) = E(x, y, \theta) = E_0 [1 + f(x, y, \theta)]; \quad f(x, y, \theta) > -1 \quad (2.2)$$

$$\rho(\mathbf{x}, \theta) = \rho(x, y, \theta) = \rho_0 [1 + g(x, y, \theta)]; \quad g(x, y, \theta) > -1 \quad (2.3)$$

$$E(\mathbf{x}, \theta) = E(x, y, z, \theta) = E_0 [1 + f(x, y, z, \theta)]; \quad f(x, y, z, \theta) > -1 \quad (2.4)$$

$$\rho(\mathbf{x}, \theta) = \rho(x, y, z, \theta) = \rho_0 [1 + g(x, y, z, \theta)]; \quad g(x, y, z, \theta) > -1 \quad (2.5)$$

The governing differential equation of motion for the beam with spatially random elastic modulus and mass density

is given by:

$$\frac{\partial^2}{\partial x^2} \left[\mathcal{D}(x, \theta) \frac{\partial^2 w(x, t, \theta)}{\partial x^2} \right] + \mathcal{M}(x, \theta) \frac{\partial^2 w(x, t, \theta)}{\partial t^2} = q(x, t) \quad (2.6)$$

where $\mathcal{D}(x, \theta)$ denotes the effective bending stiffness, defined as: $\mathcal{D}(x, \theta) = \int_A y^2 E(\mathbf{x}, \theta) dydz$ and $\mathcal{M}(x, \theta)$ represents the effective mass per unit length, given by: $\mathcal{M}(x, \theta) = \int_A \rho(\mathbf{x}, \theta) dydz$. Obtaining exact analytical solutions for the differential equations (2.6) presents significant mathematical challenges because the equation coefficients are spatial random fields with infinite degrees of freedom

2.2. Development of the Stochastic Finite Element Method via the Weighted Integral Method for Representing Spatially Random Material Properties

2.2.1. Representation of the Two-Dimensional (2D) Random Elastic Modulus Field

In the Finite Element Method (FEM) framework, the beam structure is discretized into N_e elements. Consequently, the elastic modulus of the e -th element is expressed as (2.7):

$$E_e(x, y) = E_0 [1 + f_e(x, y)]; \quad f_e(x, y) > -1 \quad (2.7)$$

This study employs a two-node beam element with two degrees of freedom (DOFs) per node. The element stiffness

matrix, $[K]_e$, is given by: $[K]_e = \int_{V_e} [B]^T [D] [B] dV_e$

$$[K]_e = \underbrace{[\Lambda]^T \left\{ \int_{V_e} y^2 E_0 [r'']^T [r''] dV_e \right\} [\Lambda]}_{[K_0]_e} + \underbrace{[\Lambda]^T \left\{ \int_{V_e} y^2 f_e(x, y) E_0 [r'']^T [r''] dV_e \right\} [\Lambda]}_{[\Delta K]_e} \quad (2.8)$$

Equation (2.8) indicates that $[K]_e$ is decomposed into two components: a deterministic part $[K_0]_e$ given by (2.9), and a stochastic part $[\Delta K]_e$ given by (2.10), expressed in terms of the random variables (2.11).

$$[K_0]_e = [\Lambda]^T \left\{ \int_{V_e} y^2 E_0 [r]^T [r] dV_e \right\} [\Lambda] = \frac{E_0 I}{L_e^3} \begin{bmatrix} 12 & 6L_e & -12 & 6L_e \\ & 4L_e^2 & -6L_e & 2L_e^2 \\ & & 12 & -6L_e \\ \text{sym.} & & & 4L_e^2 \end{bmatrix} \quad (2.9)$$

$$[\Delta K]_e = [\Lambda]^T \left\{ \int_{V_e} y^2 f_e(x, y) E_0 [r]^T [r] dV_e \right\} [\Lambda] = [\Delta K_1]_e \Theta_1^e + [\Delta K_2]_e \Theta_2^e + [\Delta K_3]_e \Theta_3^e \quad (2.10)$$

$$\Theta_i^e = \int_{A_e} x^{i-1} y^2 f_e(x, y) dA_e, i = 1, 2, 3 \quad (2.11)$$

$$[\Delta K_1]_e = \frac{E_0 b}{L_e^6} \begin{bmatrix} 36L_e^2 & 24L_e^3 & -36L_e^2 & 12L_e^3 \\ & 16L_e^4 & -24L_e^3 & 8L_e^4 \\ & & 36L_e^2 & -12L_e^3 \\ \text{sym.} & & & 4L_e^4 \end{bmatrix} \quad (2.12)$$

$$[\Delta K_2]_e = \frac{E_0 b}{L_e^6} \begin{bmatrix} -144L_e & -84L_e^2 & 144L_e & -60L_e^2 \\ & -48L_e^3 & 84L_e^2 & -36L_e^3 \\ & & -144L_e & 60L_e^2 \\ \text{sym.} & & & -24L_e^3 \end{bmatrix}; [\Delta K_3]_e = \frac{E_0 b}{L_e^6} \begin{bmatrix} 144 & 72L_e & -144 & 72L_e \\ & 36L_e^2 & -72L_e & 36L_e^2 \\ & & 144 & -72L_e \\ \text{sym.} & & & 36L_e^2 \end{bmatrix} \quad (2.13)$$

2.2.2. Representation of the Two-Dimensional (2D) Random Mass Density Field

In the FEM framework, the element mass matrix, $[M]_e$, is given by Equation (2.14):

$$[M]_e = \int_{V_e} \rho [N]^T [N] dV_e \quad (2.14)$$

As the beam structure is discretized into N_e elements, the two-dimensional (2D) random mass density is given by (2.15), where ρ_0 is the mean value of ρ , and $g(x, y)$ is a 2D zero-mean stationary Gaussian random field.

$$\rho(x, y) = \rho_0 [1 + g_e(x, y)]; \quad g_e(x, y) > -1 \quad (2.15)$$

The element mass matrix is decomposed into a deterministic part and a stochastic part:

$$[M]_e = [\Lambda]^T \underbrace{\int_{V_e} \rho_0 [r]^T [r] dV_e}_{[M_0]_e} [\Lambda] + [\Lambda]^T \underbrace{\int_{V_e} \rho_0 g_e(x, y) [r]^T [r] dV_e}_{\sum_{i=1}^7 [\Delta M_i]_e \Psi_i^e} [\Lambda] = [M_0]_e + \sum_{i=1}^7 [\Delta M_i]_e \Psi_i^e \quad (2.16)$$

$$[M_0]_e = \rho_0 A_e \begin{bmatrix} \frac{13}{35} L_e & \frac{11}{210} L_e^2 & \frac{9}{70} L_e & -\frac{13}{420} L_e^2 \\ & \frac{1}{105} L_e^3 & \frac{13}{420} L_e^2 & -\frac{1}{420} L_e^3 \\ & & \frac{13}{35} L_e & -\frac{11}{210} L_e^2 \\ \text{sym.} & & & \frac{1}{105} L_e^2 \end{bmatrix} \quad (2.17); \quad [\Delta M_1]_e = b \rho_0 \begin{bmatrix} 1 & 0 & 0 & 0 \\ & 0 & 0 & 0 \\ & & 0 & 0 \\ \text{sym.} & & & 0 \end{bmatrix} \quad (2.18)$$

$$[\Delta M_2]_e = b \rho_0 \begin{bmatrix} 0 & 1 & 0 & 0 \\ & 0 & 0 & 0 \\ & & 0 & 0 \\ \text{sym.} & & & 0 \end{bmatrix} \quad (2.19); \quad [\Delta M_3]_e = \frac{b \rho_0}{L_e^2} \begin{bmatrix} -6 & -2L_e & 3 & -L_e \\ & L_e^2 & 0 & 0 \\ & & 0 & 0 \\ \text{sym.} & & & 0 \end{bmatrix} \quad (2.20)$$

$$[\Delta M_4]_e = \frac{b \rho_0}{L_e^3} \begin{bmatrix} 4 & -2L_e & -2 & L_e \\ & -4L_e^2 & 3L_e & -L_e^2 \\ & & 0 & 0 \\ \text{sym.} & & & 0 \end{bmatrix} \quad (2.21); \quad [\Delta M_5]_e = \frac{b \rho_0}{L_e^4} \begin{bmatrix} 9 & 8L_e & -9 & 3L_e \\ & 6L_e^2 & -8L_e & 3L_e^2 \\ & & 9 & -3L_e \\ \text{sym.} & & & L_e^2 \end{bmatrix} \quad (2.22)$$

$$[\Delta M_6]_e = \frac{b \rho_0}{L_e^5} \begin{bmatrix} -12 & -7L_e & 12 & -5L_e \\ & -4L_e^2 & 7L_e & -3L_e^2 \\ & & -12 & 5L_e \\ \text{sym.} & & & -2L_e^2 \end{bmatrix} \quad (2.23); \quad [\Delta M_7]_e = \frac{b \rho_0}{L_e^6} \begin{bmatrix} 4 & 2L_e & -4 & 2L_e \\ & L_e^2 & -2L_e & L_e^2 \\ & & 4 & -2L_e \\ \text{sym.} & & & L_e^2 \end{bmatrix} \quad (2.24)$$

The random variables Ψ_i^e are defined as (2.25) via the weighted integral of the 2D mass density random field and the monomials: $\Psi_i^e = \int_{A_e} x^{i-1} g_e(x, y) dx dy \quad ; \quad i = 1, 2, 3, \dots, 7$ (2.25)

2.2.3. Representation of the Three-Dimensional (3D) Random Elastic Modulus Field

The 3D random elastic modulus of the beam is defined as (2.4), where E_0 is the mean value, and $f(x,y,z)$ represents a 3D zero-mean stationary random field. The elastic modulus of the e -th element is thus expressed as:

$$E_e(x, y, z) = E_0 [1 + f_e(x, y, z)]; \quad f_e(x, y, z) > -1 \quad (2.26)$$

The element stiffness matrix $[K]_e$ considering the 3D spatially random elastic modulus is given by (2.27), where the basic random variables are defined as (2.28), and the matrices are specified in Equations (2.29) - (2.31).

$$[K]_e = [K_0]_e + [\Delta K_1]_e \Theta_1^e + [\Delta K_2]_e \Theta_2^e + [\Delta K_3]_e \Theta_3^e \quad (2.27)$$

$$\Theta_i^e = \int_{V_e} x^{i-1} y^2 f_e(x, y, z) dV_e; \quad i = 1, 2, 3 \quad (2.28); \quad [\Delta K_1]_e = \frac{E_0}{L_e^6} \begin{bmatrix} 36L_e^2 & 24L_e^3 & -36L_e^2 & 12L_e^3 \\ 16L_e^4 & -24L_e^3 & 8L_e^4 & \\ & 36L_e^2 & -12L_e^3 & \\ \text{sym.} & & & 4L_e^4 \end{bmatrix} \quad (2.29)$$

$$[\Delta K_2]_e = \frac{E_0}{L_e^6} \begin{bmatrix} -144L_e & -84L_e^2 & 144L_e & -60L_e^2 \\ & -48L_e^3 & 84L_e^2 & -36L_e^3 \\ & & -144L_e & 60L_e^2 \\ \text{sym.} & & & -24L_e^3 \end{bmatrix} \quad (2.30); \quad [\Delta K_3]_e = \frac{E_0}{L_e^6} \begin{bmatrix} 144 & 72L_e & -144 & 72L_e \\ & 36L_e^2 & -72L_e & 36L_e^2 \\ & & 144 & -72L_e \\ \text{sym.} & & & 36L_e^2 \end{bmatrix} \quad (2.31)$$

2.2.4. Representation of the Three-Dimensional (3D) Random Mass Density Field

The 3D spatial random mass density of the beam is defined as in Equations (2.32), where $g(x,y,z)$ represents a 3D zero-mean stationary random field.

$$\rho(x, y, z) = \rho_0 [1 + g(x, y, z)]; \quad g(x, y, z) > -1 \quad (2.32)$$

The element mass matrix considering the 3D random mass density is given by Equation (2.33), where the coefficient matrices are defined in Equations (2.34) - (2.40), and the random variables are specified in Equation (2.41).

$$[M]_e = [M_0]_e + \sum_{i=1}^7 [\Delta M_i]_e \Psi_i^e \quad (2.33); \quad [\Delta M_1]_e = \rho_0 \begin{bmatrix} 1 & 0 & 0 & 0 \\ & 0 & 0 & 0 \\ & & 0 & 0 \\ \text{sym.} & & & 0 \end{bmatrix} \quad (2.34); \quad [\Delta M_2]_e = \rho_0 \begin{bmatrix} 0 & 1 & 0 & 0 \\ & 0 & 0 & 0 \\ & & 0 & 0 \\ \text{sym.} & & & 0 \end{bmatrix} \quad (2.35)$$

$$[\Delta M_3]_e = \frac{\rho_0}{L_e^2} \begin{bmatrix} -6 & -2L_e & 3 & -L_e \\ & L_e^2 & 0 & 0 \\ & & 0 & 0 \\ \text{sym.} & & & 0 \end{bmatrix} \quad (2.36); \quad [\Delta M_4]_e = \frac{\rho_0}{L_e^3} \begin{bmatrix} 4 & -2L_e & -2 & L_e \\ & -4L_e^2 & 3L_e & -L_e^2 \\ & & 0 & 0 \\ \text{sym.} & & & 0 \end{bmatrix} \quad (2.37)$$

$$[\Delta M_5]_e = \frac{\rho_0}{L_e^4} \begin{bmatrix} 9 & 8L_e & -9 & 3L_e \\ & 6L_e^2 & -8L_e & 3L_e^2 \\ & & 9 & -3L_e \\ \text{sym.} & & & L_e^2 \end{bmatrix} \quad (2.38); \quad [\Delta M_6]_e = \frac{\rho_0}{L_e^5} \begin{bmatrix} -12 & -7L_e & 12 & -5L_e \\ & -4L_e^2 & 7L_e & -3L_e^2 \\ & & -12 & 5L_e \\ \text{sym.} & & & -2L_e^2 \end{bmatrix} \quad (2.39)$$

$$[\Delta M_7]_e = \frac{\rho_0}{L_e^6} \begin{bmatrix} 4 & 2L_e & -4 & 2L_e \\ & L_e^2 & -2L_e & L_e^2 \\ & & 4 & -2L_e \\ \text{sym.} & & & L_e^2 \end{bmatrix} \quad (2.40); \quad \Psi_i^e = \int_{V_e} x^{i-1} g_e(x, y, z) dx dy dz; \quad i = 1, 2, 3, \dots, 7 \quad (2.41)$$

2.3. Development of the Stochastic Finite Element Method for Static Analysis

The global structural equilibrium equation for static analysis is given by Eq. (2.42):

$$[K]\{U\} = \{P\} \quad (2.42)$$

The global stiffness matrix is decomposed into a deterministic part $[K_0] = \sum_{e=1}^{N_e} [K_0]_e^{Ext}$ and a stochastic part that depends on the basic random variables, as given by (2.43).

$$[K] = [K_0] + \sum_{e=1}^{N_e} ([\Delta K_1]_e^{Ext} \Theta_1^e + [\Delta K_2]_e^{Ext} \Theta_2^e + [\Delta K_3]_e^{Ext} \Theta_3^e) \quad (2.43)$$

Expanding the terms in Equation (2.42) via a Taylor series about the mean values $\bar{\Theta}_i^e$ of the random variables

$\{\Theta\} = \{\Theta_i^e\}_{i=1,2,3}^{e=1,2,\dots,N_e}$ yields Equations (2.44) - (2.45).

$$[K] \approx [K_0] + \sum_{e=1}^{N_e} \sum_{i=1}^3 [K_i^e]_I (\Theta_i^e - \bar{\Theta}_i^e) + \frac{1}{2} \sum_{e_1=1}^{N_e} \sum_{e_2=1}^{N_e} \sum_{i_1=1}^3 \sum_{i_2=1}^3 [K_{i_1 i_2}^{e_1 e_2}]_{II} (\Theta_{i_1}^{e_1} - \bar{\Theta}_{i_1}^{e_1}) (\Theta_{i_2}^{e_2} - \bar{\Theta}_{i_2}^{e_2}) + \dots \quad (2.44)$$

$$\{U\} \approx \{U_0\} + \sum_{e=1}^{N_e} \sum_{i=1}^3 \{U_i^e\}_I (\Theta_i^e - \bar{\Theta}_i^e) + \frac{1}{2} \sum_{e_1=1}^{N_e} \sum_{e_2=1}^{N_e} \sum_{i_1=1}^3 \sum_{i_2=1}^3 \{U_{i_1 i_2}^{e_1 e_2}\}_{II} (\Theta_{i_1}^{e_1} - \bar{\Theta}_{i_1}^{e_1}) (\Theta_{i_2}^{e_2} - \bar{\Theta}_{i_2}^{e_2}) + \dots \quad (2.45)$$

Các ma trận xuất hiện trong (2.44) - (2.45) được xác định như (2.46).

$$\left[K_i^e \right]_I = \frac{\partial [K]}{\partial \Theta_i^e} \Big|_{\bar{\Theta}_i^e}; \left[K_{i_1 i_2}^{e_1 e_2} \right]_{II} = \frac{\partial^2 [K]}{\partial \Theta_{i_1}^{e_1} \partial \Theta_{i_2}^{e_2}} \Big|_{\bar{\Theta}_i^e}; \{U_i^e\}_I = \frac{\partial \{U\}}{\partial \Theta_i^e} \Big|_{\bar{\Theta}_i^e}; \{U_{i_1 i_2}^{e_1 e_2}\}_{II} = \frac{\partial^2 \{U\}}{\partial \Theta_{i_1}^{e_1} \partial \Theta_{i_2}^{e_2}} \Big|_{\bar{\Theta}_i^e} \quad (2.46)$$

In Equation (2.46), $[\bullet]_I$ and $[\bullet]_{II}$ represent the first and second partial derivatives with respect to the basic random variables, evaluated at their mean values. Because the random variables Θ_i^e are discretized from the same zero-mean random field $f(x,y)$ or $f(x,y,z)$, we have $\bar{\Theta}_i^e = 0$. Substituting these into Equations (2.44) - (2.45) gives:

$$[K] \approx [K_0] + \sum_{e=1}^{N_e} \sum_{i=1}^3 [K_i^e]_I \Theta_i^e + \frac{1}{2} \sum_{e_1=1}^{N_e} \sum_{e_2=1}^{N_e} \sum_{i_1=1}^3 \sum_{i_2=1}^3 [K_{i_1 i_2}^{e_1 e_2}]_{II} \Theta_{i_1}^{e_1} \Theta_{i_2}^{e_2} + \dots \quad (2.47)$$

$$\{U\} \approx \{U_0\} + \sum_{e=1}^{N_e} \sum_{i=1}^3 \{U_i^e\}_I \Theta_i^e + \frac{1}{2} \sum_{e_1=1}^{N_e} \sum_{e_2=1}^{N_e} \sum_{i_1=1}^3 \sum_{i_2=1}^3 \{U_{i_1 i_2}^{e_1 e_2}\}_{II} \Theta_{i_1}^{e_1} \Theta_{i_2}^{e_2} + \dots \quad (2.48)$$

Substituting Equations (2.47) - (2.48) into Equation (2.42) and equating the coefficients of the partial derivatives of the same order yields the partial derivatives of the displacement vector. Consequently, the first-order approximation of the global displacement vector $\{U\}$ is given by (2.49).

$$\{U\} \approx \{U_0\} - \sum_{e=1}^{N_e} \sum_{i=1}^3 [K_0]^{-1} [K_i^e]_I \{U_0\} \Theta_i^e \quad (2.49)$$

Applying the expectation and variance operators yields the expected value and covariance matrix of the nodal displacements, which are given by Equations (2.50) - (2.51).

$$\mathbb{E}[\{U\}] \approx \mathbb{E} \left[\{U_0\} - \sum_{e=1}^{N_e} \sum_{i=1}^3 [K_0]^{-1} [K_i^e]_I \{U_0\} \Theta_i^e \right] \approx \{U_0\} \quad (2.50)$$

$$\text{Cov}[\{U\}, \{U\}^T] = \sum_{e_1=1}^{N_e} \sum_{e_2=1}^{N_e} \sum_{i_1=1}^3 \sum_{i_2=1}^3 \left\{ [K_0]^{-1} [K_{i_1}^{e_1}]_I \{U_0\} \{U_0\}^T \left([K_{i_2}^{e_2}]_I \right)^T \left([K_0]^{-1} \right)^T \right\} \mathbb{E}[\Theta_{i_1}^{e_1} \Theta_{i_2}^{e_2}] \quad (2.51)$$

The variances of the nodal displacements considering the 2D, 3D random elastic moduli are given respectively by:

$$\text{Var}[\{U\}, \{U\}^T] = \sum_{e_1=1}^{N_e} \sum_{e_2=1}^{N_e} \sum_{i_1=1}^3 \sum_{i_2=1}^3 \left\{ \text{diag} \left([K_0]^{-1} [\Delta K_{i_1}^{e_1}]_e^{\text{Ext}} \{U_0\} \right) \{U_0\}^T \left([\Delta K_{i_2}^{e_2}]_e^{\text{Ext}} \right)^T \left([K_0]^{-1} \right)^T \right\} \int_{A_{e_1}} \int_{A_{e_2}} x_1^{i_1-1} y_1^2 x_2^{i_2-1} y_2^2 R_{f_{e_1} f_{e_2}} dA_{e_1} dA_{e_2} \quad (2.52)$$

$$\text{Var}[\{U\}, \{U\}^T] = \sum_{e_1=1}^{N_e} \sum_{e_2=1}^{N_e} \sum_{i_1=1}^3 \sum_{i_2=1}^3 \left\{ \text{diag} \left([K_0]^{-1} [\Delta K_{i_1}^{e_1}]_e^{\text{Ext}} \{U_0\} \right) \{U_0\}^T \left([\Delta K_{i_2}^{e_2}]_e^{\text{Ext}} \right)^T \left([K_0]^{-1} \right)^T \right\} \int_{V_{e_1}} \int_{V_{e_2}} x_1^{i_1-1} y_1^2 x_2^{i_2-1} y_2^2 R_{f_{e_1} f_{e_2}} dV_{e_1} dV_{e_2} \quad (2.53)$$

The COV of the mid-span nodal displacement is given by:

$$\text{COV} = \frac{\sqrt{\text{Var}[\{U\}, \{U\}^T]_{\frac{L}{2}}}}{\left| \mathbb{E}[\{U\}]_{\frac{L}{2}} \right|} \quad (2.54)$$

2.4. Development of the Stochastic Finite Element Method for Linear Dynamic Characteristic Analysis

2.4.1. Free Bending Vibration Analysis of the Beam by the Finite Element Method

Consider a rectangular beam with height h , width b , and span L . The undamped free vibration equation of the beam is expressed as:

$$([K] - \omega^2 [M]) \{\Phi\} = 0 \quad (2.55)$$

The natural angular frequency is given by Equation (2.56):

$$\text{Det}([K] - \omega^2 [M]) = 0 \quad (2.56)$$

2.4.2. Development of the Stochastic Finite Element Method for the Analysis of Linear Dynamic Characteristics of Beams with Spatially Random Material Properties

The stiffness and mass matrices of the beam structure, considering the 2D and 3D spatially random mass density, are determined as follows:

$$[K] = \sum_{e=1}^{N_e} [K]_e = \sum_{e=1}^{N_e} [K_0]_e^{Ext} + \sum_{e=1}^{N_e} \sum_{i=1}^3 [\Delta K_i]_e^{Ext} \Theta_i^e = [K_0] + \sum_{e=1}^{N_e} \sum_{i=1}^3 [\Delta K_i]_e^{Ext} \Theta_i^e \quad (2.57)$$

$$[M] = \sum_{e=1}^{N_e} [M]_e = \sum_{e=1}^{N_e} [M_0]_e^{Ext} + \sum_{e=1}^{N_e} \sum_{i=1}^7 [\Delta M_i]_e^{Ext} \Psi_i^e = [M_0] + \sum_{e=1}^{N_e} \sum_{i=1}^7 [\Delta M_i]_e^{Ext} \Psi_i^e \quad (2.58)$$

For the k -th eigenvalue and the k -th mode shape:

$$\left[\underbrace{\left([K_0] + \sum_{e=1}^{N_e} \sum_{i=1}^3 [\Delta K_i]_e^{Ext} \Theta_i^e \right)}_{[K]} - \lambda_k \left([M_0] + \sum_{e=1}^{N_e} \sum_{i=1}^7 [\Delta M_i]_e^{Ext} \Psi_i^e \right) \right] \{\Phi\}_k = 0 \quad (2.59)$$

Expanding the stiffness and the mass matrix, the k -th eigenvalue, and the k -th mode shape via a Taylor series yields:

$$[K] \approx [K_0] + \sum_{e=1}^{N_e} \sum_{i=1}^3 \left[\frac{\partial [K]}{\partial \Theta_i^e} \right]_{\bar{\Theta}_i^e} \left(\Theta_i^e - \bar{\Theta}_i^e \right) + \frac{1}{2} \sum_{e_1=1}^{N_e} \sum_{e_2=1}^{N_e} \sum_{i_1=1}^3 \sum_{i_2=1}^3 \left[\frac{\partial^2 [K]}{\partial \Theta_{i_1}^{e_1} \partial \Theta_{i_2}^{e_2}} \right]_{\bar{\Theta}_i^e} \left(\Theta_{i_1}^{e_1} - \bar{\Theta}_{i_1}^{e_1} \right) \left(\Theta_{i_2}^{e_2} - \bar{\Theta}_{i_2}^{e_2} \right) + \dots \quad (2.60)$$

$$[M] \approx [M_0] + \sum_{e=1}^{N_e} \sum_{i=1}^7 \left[\frac{\partial [M]}{\partial \Psi_i^e} \right]_{\bar{\Psi}_i^e} \left(\Psi_i^e - \bar{\Psi}_i^e \right) + \frac{1}{2} \sum_{e_1=1}^{N_e} \sum_{e_2=1}^{N_e} \sum_{i_1=1}^7 \sum_{i_2=1}^7 \left[\frac{\partial^2 M}{\partial \Psi_{i_1}^{e_1} \partial \Psi_{i_2}^{e_2}} \right]_{\bar{\Psi}_i^e} \left(\Psi_{i_1}^{e_1} - \bar{\Psi}_{i_1}^{e_1} \right) \left(\Psi_{i_2}^{e_2} - \bar{\Psi}_{i_2}^{e_2} \right) + \dots \quad (2.61)$$

$$\lambda_k \approx \lambda_k^0 + \sum_{e=1}^{N_e} \sum_{i=1}^3 \left[\frac{\partial \lambda_k}{\partial \Theta_i^e} \right]_{\bar{\Theta}_i^e} \left(\Theta_i^e - \bar{\Theta}_i^e \right) + \sum_{e=1}^{N_e} \sum_{i=1}^7 \left[\frac{\partial \lambda_k}{\partial \Psi_i^e} \right]_{\bar{\Psi}_i^e} \left(\Psi_i^e - \bar{\Psi}_i^e \right) + \frac{1}{2} \sum_{e_1=1}^{N_e} \sum_{e_2=1}^{N_e} \sum_{i_1=1}^3 \sum_{i_2=1}^3 \left[\frac{\partial^2 \lambda_k}{\partial \Theta_{i_1}^{e_1} \partial \Theta_{i_2}^{e_2}} \right]_{\bar{\Theta}_i^e} \left(\Theta_{i_1}^{e_1} - \bar{\Theta}_{i_1}^{e_1} \right) \left(\Theta_{i_2}^{e_2} - \bar{\Theta}_{i_2}^{e_2} \right) \quad (2.62)$$

$$+ \frac{1}{2} \sum_{e_1=1}^{N_e} \sum_{e_2=1}^{N_e} \sum_{i_1=1}^7 \sum_{i_2=1}^7 \left[\frac{\partial^2 \lambda_k}{\partial \Psi_{i_1}^{e_1} \partial \Psi_{i_2}^{e_2}} \right]_{\bar{\Psi}_i^e} \left(\Psi_{i_1}^{e_1} - \bar{\Psi}_{i_1}^{e_1} \right) \left(\Psi_{i_2}^{e_2} - \bar{\Psi}_{i_2}^{e_2} \right) + \frac{1}{2} \sum_{e_1=1}^{N_e} \sum_{e_2=1}^{N_e} \sum_{i_1=1}^3 \sum_{i_2=1}^3 \left[\frac{\partial^2 \lambda_k}{\partial \Theta_{i_1}^{e_1} \partial \Psi_{i_2}^{e_2}} \right]_{\bar{\Theta}_i^e, \bar{\Psi}_i^e} \left(\Theta_{i_1}^{e_1} - \bar{\Theta}_{i_1}^{e_1} \right) \left(\Psi_{i_2}^{e_2} - \bar{\Psi}_{i_2}^{e_2} \right) + \dots$$

$$\{\Phi\}_k \approx \{\Phi\}_k^0 + \sum_{e=1}^{N_e} \sum_{i=1}^3 \left[\frac{\partial \{\Phi\}_k}{\partial \Theta_i^e} \right]_{\bar{\Theta}_i^e} \left(\Theta_i^e - \bar{\Theta}_i^e \right) + \sum_{e=1}^{N_e} \sum_{i=1}^7 \left[\frac{\partial \{\Phi\}_k}{\partial \Psi_i^e} \right]_{\bar{\Psi}_i^e} \left(\Psi_i^e - \bar{\Psi}_i^e \right) + \frac{1}{2} \sum_{e_1=1}^{N_e} \sum_{e_2=1}^{N_e} \sum_{i_1=1}^3 \sum_{i_2=1}^3 \left[\frac{\partial^2 \{\Phi\}_k}{\partial \Theta_{i_1}^{e_1} \partial \Theta_{i_2}^{e_2}} \right]_{\bar{\Theta}_i^e} \left(\Theta_{i_1}^{e_1} - \bar{\Theta}_{i_1}^{e_1} \right) \left(\Theta_{i_2}^{e_2} - \bar{\Theta}_{i_2}^{e_2} \right) \quad (2.63)$$

$$+ \frac{1}{2} \sum_{e_1=1}^{N_e} \sum_{e_2=1}^{N_e} \sum_{i_1=1}^7 \sum_{i_2=1}^7 \left[\frac{\partial^2 \{\Phi\}_k}{\partial \Psi_{i_1}^{e_1} \partial \Psi_{i_2}^{e_2}} \right]_{\bar{\Psi}_i^e} \left(\Psi_{i_1}^{e_1} - \bar{\Psi}_{i_1}^{e_1} \right) \left(\Psi_{i_2}^{e_2} - \bar{\Psi}_{i_2}^{e_2} \right) + \frac{1}{2} \sum_{e_1=1}^{N_e} \sum_{e_2=1}^{N_e} \sum_{i_1=1}^3 \sum_{i_2=1}^3 \left[\frac{\partial^2 \{\Phi\}_k}{\partial \Theta_{i_1}^{e_1} \partial \Psi_{i_2}^{e_2}} \right]_{\bar{\Theta}_i^e, \bar{\Psi}_i^e} \left(\Theta_{i_1}^{e_1} - \bar{\Theta}_{i_1}^{e_1} \right) \left(\Psi_{i_2}^{e_2} - \bar{\Psi}_{i_2}^{e_2} \right) + \dots$$

The first-order approximation of the eigenvalue is given by the following equation:

$$\lambda_k \approx \lambda_k^0 + \sum_{e=1}^{N_e} \sum_{i=1}^3 \left\{ \Phi_k^0 \right\}^T [\Delta K_i]_e^{Ext} \left\{ \Phi_k^0 \right\} \Theta_i^e + \sum_{e=1}^{N_e} \sum_{i=1}^7 \left(- \left\{ \Phi_k^0 \right\}^T \lambda_k^0 [\Delta M_i]_e^{Ext} \left\{ \Phi_k^0 \right\} \right) \Psi_i^e \quad (2.64)$$

The expected value of the k -th eigenvalue is given by the following equation:

$$\mathbb{E}[\lambda_k] \approx \mathbb{E} \left[\lambda_k^0 + \sum_{e=1}^{N_e} \sum_{i=1}^3 \left\{ \Phi_k^0 \right\}^T [\Delta K_i]_e^{Ext} \left\{ \Phi_k^0 \right\} \Theta_i^e + \sum_{e=1}^{N_e} \sum_{i=1}^7 \left(- \left\{ \Phi_k^0 \right\}^T \lambda_k^0 [\Delta M_i]_e^{Ext} \left\{ \Phi_k^0 \right\} \right) \Psi_i^e \right] \approx \lambda_k^0 \quad (2.65)$$

The variance of the k -th eigenvalue in the case of two-dimensional (2D) random material properties is given by:

$$\begin{aligned} Var[\lambda_k] &= \mathbb{E} \left[\left(\lambda_k - \mathbb{E}[\lambda_k] \right) \left(\lambda_k - \mathbb{E}[\lambda_k] \right) \right] \\ &= \sum_{e_1=1}^{N_e} \sum_{e_2=1}^{N_e} \sum_{i_1=1}^3 \sum_{i_2=1}^3 \left\{ \left\{ \Phi_k^0 \right\}^T [\Delta K_{i_1}]_{e_1}^{Ext} \left\{ \Phi_k^0 \right\} \left\{ \Phi_k^0 \right\}^T [\Delta K_{i_2}]_{e_2}^{Ext} \left\{ \Phi_k^0 \right\} \int_{A_{e_1}} \int_{A_{e_2}} x_1^{i_1-1} y_1^{i_1-1} x_2^{i_2-1} y_2^{i_2-1} R_{ff} dA_{e_1} dA_{e_2} \right\} \\ &\quad + \sum_{e_1=1}^{N_e} \sum_{e_2=1}^{N_e} \sum_{i_1=1}^7 \sum_{i_2=1}^7 \left\{ \left\{ \Phi_k^0 \right\}^T \lambda_k^0 [\Delta M_{i_1}]_{e_1}^{Ext} \left\{ \Phi_k^0 \right\} \left\{ \Phi_k^0 \right\}^T \lambda_k^0 [\Delta M_{i_2}]_{e_2}^{Ext} \left\{ \Phi_k^0 \right\} \int_{A_{e_1}} \int_{A_{e_2}} x_1^{i_1-1} x_2^{i_2-1} R_{gg} dA_{e_1} dA_{e_2} \right\} \\ &\quad + 2 \sum_{e_1=1}^{N_e} \sum_{e_2=1}^{N_e} \sum_{i_1=1}^3 \sum_{i_2=1}^7 \left\{ \left\{ \Phi_k^0 \right\}^T [\Delta K_{i_1}]_{e_1}^{Ext} \left\{ \Phi_k^0 \right\} \left\{ \Phi_k^0 \right\}^T \left\{ \Phi_k^0 \right\}^T (-\lambda_k^0) [\Delta M_{i_2}]_{e_2}^{Ext} \left\{ \Phi_k^0 \right\} \int_{A_{e_1}} \int_{A_{e_2}} x_1^{i_1-1} y_1^{i_1-1} x_2^{i_2-1} R_{fg} dA_{e_1} dA_{e_2} \right\} \end{aligned} \quad (2.66)$$

For the case of three-dimensional (3D) random E and ρ , the variance of the k -th eigenvalue is given by:

$$\begin{aligned}
 \text{Var}[\lambda_k] &= \mathbb{E} \left[\left(\lambda_k - \mathbb{E}[\lambda_k] \right) \left(\lambda_k - \mathbb{E}[\lambda_k] \right)^T \right] \\
 &= \sum_{e_1=1}^{N_e} \sum_{e_2=1}^{N_e} \sum_{i_1=1}^3 \sum_{i_2=1}^3 \left\{ \left\{ \Phi_k^0 \right\}^T \left[\Delta K_{i_1} \right]_e^{\text{Ext}} \left\{ \Phi_k^0 \right\} \left\{ \Phi_k^0 \right\}^T \left[\Delta K_{i_2} \right]_e^{\text{Ext}} \left\{ \Phi_k^0 \right\} \times \int_{V_{e_1}} \int_{V_{e_2}} x_1^{i_1-1} y_1^2 x_2^{i_2-1} y_2^2 R_{ff} dV_{e_1} dV_{e_2} \right\} \\
 &+ \sum_{e_1=1}^{N_e} \sum_{e_2=1}^{N_e} \sum_{i_1=1}^7 \sum_{i_2=1}^7 \left\{ \left\{ \Phi_k^0 \right\}^T \lambda_k^0 \left[\Delta M_{i_1} \right]_e^{\text{Ext}} \left\{ \Phi_k^0 \right\} \left\{ \Phi_k^0 \right\}^T \lambda_k^0 \left[\Delta M_{i_2} \right]_e^{\text{Ext}} \left\{ \Phi_k^0 \right\} \times \int_{V_{e_1}} \int_{V_{e_2}} x_1^{i_1-1} x_2^{i_2-1} R_{gg} dV_{e_1} dV_{e_2} \right\} \\
 &+ 2 \sum_{e_1=1}^{N_e} \sum_{e_2=1}^{N_e} \sum_{i_1=1}^3 \sum_{i_2=1}^7 \left\{ \left\{ \Phi_k^0 \right\}^T \left[\Delta K_{i_1} \right]_e^{\text{Ext}} \left\{ \Phi_k^0 \right\} \left\{ \Phi_k^0 \right\}^T \left\{ \Phi_k^0 \right\}^T \left(-\lambda_k^0 \right) \left[\Delta M_{i_2} \right]_e^{\text{Ext}} \left\{ \Phi_k^0 \right\} \times \int_{V_{e_1}} \int_{V_{e_2}} x_1^{i_1-1} y_1^2 x_2^{i_2-1} R_{fg} dV_{e_1} dV_{e_2} \right\}
 \end{aligned} \tag{2.67}$$

$$\text{The COV of the eigenvalue is used to compare different data series: } \text{COV} = \frac{\sqrt{\text{Var}[\lambda_k]}}{\mathbb{E}[\lambda_k]} \tag{2.68}$$

2.5. Response Analysis of Beams with Spatially Random Material Properties Using Monte Carlo Simulation

2.5.1. Generation of 2D Material Property Random Fields Using the Spectral Method

Using the spectral representation method [107], [108] the random field is expressed as a trigonometric series:

$$f(x, y) = \sqrt{2} \sum_{n_1=0}^{N_1-1} \sum_{n_2=0}^{N_2-1} \left\{ A_{n_1 n_2}^f \cos \left[\kappa_{1n_1}^f x + \kappa_{2n_2}^f y + \Phi_{n_1 n_2}^{(1f)} \right] + B_{n_1 n_2}^f \cos \left[\kappa_{1n_1}^f x - \kappa_{2n_2}^f y + \Phi_{n_1 n_2}^{(2f)} \right] \right\} \tag{2.69}$$

The mass density random field is expressed as follows:

$$\begin{aligned}
 g(x, y) &= \gamma \left(\frac{\sigma_f}{\sigma_g} \right) \sqrt{2} \sum_{n_1=0}^{N_1-1} \sum_{n_2=0}^{N_2-1} \left\{ A_{n_1 n_2}^f \cos \left[\kappa_{1n_1}^f x + \kappa_{2n_2}^f y + \Phi_{n_1 n_2}^{(1f)} \right] + B_{n_1 n_2}^f \cos \left[\kappa_{1n_1}^f x - \kappa_{2n_2}^f y + \Phi_{n_1 n_2}^{(2f)} \right] \right\} \\
 &+ \sqrt{1-\gamma^2} \sqrt{2} \sum_{n_1=0}^{N_1-1} \sum_{n_2=0}^{N_2-1} \left\{ A_{n_1 n_2}^g \cos \left[\kappa_{1n_1}^g x + \kappa_{2n_2}^g y + \Phi_{n_1 n_2}^{(1g)} \right] + B_{n_1 n_2}^g \cos \left[\kappa_{1n_1}^g x - \kappa_{2n_2}^g y + \Phi_{n_1 n_2}^{(2g)} \right] \right\}
 \end{aligned} \tag{2.70}$$

2.5.2. Generation of 3D Material Property Random Fields Using the Spectral Method

The univariate, three-dimensional, stationary Gaussian random field is expressed as:

$$f(x, y, z) = \sqrt{2} \sum_{n_1=0}^{N_1-1} \sum_{n_2=0}^{N_2-1} \sum_{n_3=0}^{N_3-1} \left\{ A_{n_1 n_2 n_3}^{(1f)} \cos \left[\kappa_{1n_1}^f x + \kappa_{2n_2}^f y + \kappa_{3n_3}^f z + \Phi_{n_1 n_2 n_3}^{(1f)} \right] + A_{n_1 n_2 n_3}^{(2f)} \cos \left[\kappa_{1n_1}^f x + \kappa_{2n_2}^f y - \kappa_{3n_3}^f z + \Phi_{n_1 n_2 n_3}^{(2f)} \right] \right. \\
 \left. + A_{n_1 n_2 n_3}^{(3f)} \cos \left[\kappa_{1n_1}^f x - \kappa_{2n_2}^f y + \kappa_{3n_3}^f z + \Phi_{n_1 n_2 n_3}^{(3f)} \right] + A_{n_1 n_2 n_3}^{(4f)} \cos \left[\kappa_{1n_1}^f x - \kappa_{2n_2}^f y - \kappa_{3n_3}^f z + \Phi_{n_1 n_2 n_3}^{(4f)} \right] \right\} \tag{2.71}$$

The mass density, elastic modulus random fields are correlated with each other through the correlation coefficient γ .

$$\begin{aligned}
 g(x, y, z) &= \gamma \left(\frac{\sigma_f}{\sigma_g} \right) \sqrt{2} \sum_{n_1=0}^{N_1-1} \sum_{n_2=0}^{N_2-1} \sum_{n_3=0}^{N_3-1} \left\{ A_{n_1 n_2 n_3}^{(1f)} \cos \left[\kappa_{1n_1}^f x + \kappa_{2n_2}^f y + \kappa_{3n_3}^f z + \Phi_{n_1 n_2 n_3}^{(1f)} \right] + A_{n_1 n_2 n_3}^{(2f)} \cos \left[\kappa_{1n_1}^f x + \kappa_{2n_2}^f y - \kappa_{3n_3}^f z + \Phi_{n_1 n_2 n_3}^{(2f)} \right] \right. \\
 &\left. + A_{n_1 n_2 n_3}^{(3f)} \cos \left[\kappa_{1n_1}^f x - \kappa_{2n_2}^f y + \kappa_{3n_3}^f z + \Phi_{n_1 n_2 n_3}^{(3f)} \right] + A_{n_1 n_2 n_3}^{(4f)} \cos \left[\kappa_{1n_1}^f x - \kappa_{2n_2}^f y - \kappa_{3n_3}^f z + \Phi_{n_1 n_2 n_3}^{(4f)} \right] \right\} \\
 &+ \sqrt{1-\gamma^2} \sqrt{2} \sum_{n_1=0}^{N_1-1} \sum_{n_2=0}^{N_2-1} \sum_{n_3=0}^{N_3-1} \left\{ A_{n_1 n_2 n_3}^{(1g)} \cos \left[\kappa_{1n_1}^g x + \kappa_{2n_2}^g y + \kappa_{3n_3}^g z + \Phi_{n_1 n_2 n_3}^{(1g)} \right] + A_{n_1 n_2 n_3}^{(2g)} \cos \left[\kappa_{1n_1}^g x + \kappa_{2n_2}^g y - \kappa_{3n_3}^g z + \Phi_{n_1 n_2 n_3}^{(2g)} \right] \right. \\
 &\left. + A_{n_1 n_2 n_3}^{(3g)} \cos \left[\kappa_{1n_1}^g x - \kappa_{2n_2}^g y + \kappa_{3n_3}^g z + \Phi_{n_1 n_2 n_3}^{(3g)} \right] + A_{n_1 n_2 n_3}^{(4g)} \cos \left[\kappa_{1n_1}^g x - \kappa_{2n_2}^g y - \kappa_{3n_3}^g z + \Phi_{n_1 n_2 n_3}^{(4g)} \right] \right\}
 \end{aligned} \tag{2.72}$$

2.5.3. Static Response Analysis of Beams with Spatially Random Material Properties Using MCS

Using Equation (2.69) or (2.70), N samples of the elastic modulus random field are generated. The global equilibrium equation of the structure is established based on the FEM as Equation (2.73), where U^i is the global nodal displacement vector of the structure in the i -th simulation

$$[K^i] \{U^i\} = \{P\} \tag{2.73}$$

By repeating the computation over the sample space of the elastic modulus random field, a dataset of the random responses of the beam is obtained. Statistical analysis of this dataset yields the statistical characteristics of the j -th nodal displacement: the sample mean, the sample variance, and the coefficient of variation:

$$\bar{U}_j = \frac{1}{N} \sum_{i=1}^N U_j^i; \quad s_{U_j}^2 = \frac{1}{N-1} \sum_{i=1}^N \left(U_j^i - \bar{U}_j \right)^2; \quad \text{COV} = \frac{\sqrt{s_{U_j}^2}}{|\bar{U}_j|} \tag{2.74}$$

2.5.4. Analysis of Linear Dynamic Characteristics of Beams with Spatially Random Material Properties Using Monte Carlo Simulation

Using Equation (2.56), the eigenvalue in the i -th simulation is determined according to (2.75)

$$\text{Det} \left[\left(\sum_{e=1}^{N_e} [K_0]_{e,i}^{\text{Ext}} + \sum_{e=1}^{N_e} [\Delta K]_{e,i}^{\text{Ext}} \right) - \lambda^i \left(\sum_{e=1}^{N_e} [M_0]_{e,i}^{\text{Ext}} + \sum_{e=1}^{N_e} [\Delta M]_{e,i}^{\text{Ext}} \right) \right] = 0 \tag{2.75}$$

Using Equation (2.75), the k -th eigenvalue in the i -th simulation is obtained as λ_k^i . Statistical analysis of the obtained dataset yields the sample mean and sample variance of the k -th eigenvalue, as given by the following equations:

$$\bar{\lambda}_k = \frac{1}{N} \sum_{i=1}^N \lambda_k^i; \quad s_{\lambda_k}^2 = \frac{1}{N-1} \sum_{i=1}^N (\lambda_k^i - \bar{\lambda}_k)^2; \quad \text{COV} = \frac{\sqrt{s_{\lambda_k}^2}}{\bar{\lambda}_k} \quad (2.76)$$

2.6. Validation of the Stochastic Finite Element Model for Response Analysis of Beams with Spatially Random Material Properties

2.6.1. Evaluation of Secondary Effects and the Basis for Applying the Planar Beam Model to Problems with 3D Random Material Properties

Quantitative analyses indicate: even if the E field has a large coefficient of variation (up to 20%), the fluctuations of secondary geometric characteristics, including the elastic centroid shifts (y_n, z_n) and the product of inertia (D_{yz}), only fluctuate at the noise level (under 5%). A validation simulation on a cantilever beam shows that the variation of secondary displacements is small (under 2%) compared to the primary bending displacement. The difference in bending response between the spatial model fully considering secondary effects and the planar beam model is under the 5% threshold. These results show that the application of a 2D beam model to a 3D random material problem is an acceptable simplification within the research scope.

2.6.2. Convergence Study of the Response Coefficient of Variation with Respect to Mesh Density

To verify the numerical stability of the proposed SFEM model, a mesh convergence investigation is conducted on a simply supported beam. E and ρ are modeled as spatial random fields with mean values $E_0 = 30 \times 10^9 \text{ N/m}^2$, $\rho_0 = 2400 \text{ kg/m}^3$, $\sigma_f = \sigma_g = 0.05$ and correlation lengths $d_x = d_y = d_z = d$, varying from $0.01L$ to $100L$. The COV of the mid-span displacement and the eigenvalues are evaluated across mesh configurations of $N_e = \{20, 30, 40\}$. The analytical results indicate that the statistical responses converge distinctly and the discrepancies practically vanish, confirming the mesh independence of the solution.

2.6.3. Verification of the Stochastic Finite Element Model Based on Stochastic Computational Theory

Based on the principles of stochastic mechanics, the system must converge to a deterministic state when material dispersion is very small. By establishing the standard deviations of the random fields at $\sigma_f = \sigma_g = 0.00001$, the analytical results verify that the COV of both the displacements and the eigenvalues decreases and asymptotes to an infinitely small (insignificant) value. This convergence to the deterministic solution is a mathematical proof that confirms the correctness, numerical stability, and accuracy of the theoretical SFEM framework.

2.6.4. Validation of the SFEM Model Against Published Research

2.6.4.1. Validation of Static Response Analysis for Beams with Spatially Random Material Properties

To verify the reliability of the proposed SFEM algorithm, this dissertation compares the stochastic static response with the study by Adhikari and Mukherjee [16] on a cantilever beam ($L = 1\text{m}$, $h = 0.03\text{m}$, $b = 0.003\text{m}$, expected elastic modulus $E_0 = 30 \times 10^9 \text{ N/m}^2$) subjected to a concentrated load at its free end. To establish compatibility conditions from the multidimensional model framework to the one-dimensional (1D) problem of the reference study, the axial correlation length is set to $d_x = L/10$. Meanwhile, the correlation lengths in the other directions are set to very large values ($d_y = d_z = 100L$) to eliminate random fluctuations over the cross-section, effectively reducing the 3D field to an asymptotically homogeneous state according to random field theory [51, 52]. Mathematically, since the displacement response is approximated via a first-order Taylor expansion of random variables discretized from a zero-mean field, the expected value of the normalized displacement always equals unity. This theoretical convergence, combined with the close agreement of the displacement standard deviation with the MCS results in [16], provides comprehensive evidence affirming the accuracy and reliability of the developed multidimensional SFEM model.

2.6.4.2. Validation of Linear Dynamic Characteristics Analysis of Beams with Spatially Random Material Properties

To verify the reliability of the multidimensional SFEM model in linear dynamic problems, this dissertation compares the COV of the first eigenvalue with the study by Hosseini and Khadem [63] on a cantilever beam ($L = 0.8\text{m}$, $b = 0.015\text{m}$, $h = 0.04\text{m}$; expected values $E_0 = 165.44 \times 10^9 \text{ N/m}^2$, $\rho_0 = 7830 \text{ kg/m}^3$). The Gaussian random field structure is established with standard deviations $\sigma_f = \sigma_g = 0.1$ and cross-correlation coefficients $\gamma = \{0, 1\}$. The axial correlation length is set to $d_x = L$ (based on the parameters in [63]), while the others are set to ($d_y = d_z = 100L$) to eliminate random variations in the transverse and vertical directions. The comparison results record a very small error of less than 5%.

2.6.5. Comparison of Beam Response Analysis Results Using the Stochastic Finite Element Method and MCS

To independently verify the reliability of the SFEM model, the dissertation applies the MCS method based on the principle of statistical convergence for large samples. Specifically, the spectral representation method is applied to generate $N = 10,000$ realizations of the spatial random fields E and ρ with standard deviations $\sigma_f = \sigma_g = 0.05$ strictly adhering to the initially established correlation structure. The response for each material property sample is solved using the standard FEM to construct a reference statistical dataset. Examining the histogram and Q-Q plot for the normalized displacement at the mid-span (in both 2D and 3D random models) shows that the sample quantiles closely follow the theoretical linear path, with skewness and kurtosis approaching a normal distribution. Despite some minor fluctuations in the tail regions, the overall displacement response can be considered approximately

Gaussian. This statistical convergence, combined with the small discrepancy in the expected value and coefficient of variation (COV) between the SFEM and MCS models, provides solid numerical empirical evidence to affirm the accuracy of the proposed SFEM

Extending to the linear dynamic characteristics analysis, the statistical analysis of the unnormalized absolute eigenvalues confirms a close convergence to the normal distribution. This convergence is evidenced by the Q-Q plot closely following the theoretical linear path, where the skewness and kurtosis approximate Gaussian characteristics with almost negligible differences in the tail regions. Furthermore, the high agreement in the expected values and COV between the SFEM and the MCS sample sets under various levels of material fluctuation demonstrates the accuracy and superiority of the proposed SFEM theoretical framework.

To comprehensively evaluate the accuracy of the SFEM algorithm, the COV of the static displacement and the eigenvalues are directly compared against the independent MCS dataset. The analysis is conducted on both 2D and 3D random material structures, incorporating a large-scale survey with correlation lengths varying from $0.01L$ to $100L$. The comparison process records a convergence between the two methods, with the error margin strictly controlled below the 5% threshold across the entire investigated domain. This quantitative similarity serves as robust proof of the reliability of the proposed SFEM theoretical framework for both static and linear dynamic characteristics analysis problems.

2.7. Conclusions of Chapter II

The formulation of the SFEM approach for analyzing the mechanical behavior of beam structures with spatially random material properties has been successfully accomplished in Chapter II. The core findings and contributions include:

- **Regarding the development of the theoretical framework and computational model:** The dissertation has successfully integrated the weighted integral technique to discretize the multidimensional random field and the perturbation method for uncertainty propagation. This approach enables the explicit formulation of the random stiffness and mass matrices, thereby directly solving the first-order approximation series of the structural response without requiring enormous iterative computational costs.
- **Based on the proposed SFEM model, two fundamental problems of stochastic mechanics for beam structures have been fully resolved in Chapter II:**
 - (i) *Stochastic static analysis:* Directly determining the statistical characteristics of the displacements.
 - (ii) *Stochastic linear dynamic characteristics analysis:* Determining the statistical characteristics of the eigenvalues.
- **Regarding the verification of reliability and accuracy:** The proposed algorithm has passed rigorous and systematic validation steps based on four criteria:
 - ✓ Ensuring numerical convergence and the mesh independence of the solution.
 - ✓ Approaching the deterministic solution as material fluctuations approach zero, strictly adhering to the physical logic of stochastic mechanics.
 - ✓ Exhibiting high agreement (error under 5%) when directly compared with previous studies.
 - ✓ Demonstrating strong consistency with large-sample-based Monte Carlo Simulations (MCS).

CHAPTER III: DEVELOPMENT OF THE STOCHASTIC FINITE ELEMENT METHOD FOR EULER BUCKLING ANALYSIS OF COLUMNS WITH SPATIALLY RANDOM MATERIAL PROPERTIES

3.1. Euler Buckling Analysis of Axially Compressed Columns Using the FEM

The governing differential equation for the stability of the column is expressed as (3.1): $E_0 I \frac{d^4 w}{dx^4} + P \frac{d^2 w}{dx^2} = 0$ (3.1)

The governing equation for the column stability analysis is presented in Equation (3.2): $([K] - \lambda [K_p])\{\psi\} = 0$ (3.2)

where: $[K] = \sum_{e=1}^{N_e} [K]_e^{Ext}$, $[K_p] = \sum_{e=1}^{N_e} [K_p]_e^{Ext}$, $\lambda = P$ is the eigenvalue corresponding to the critical load, $\{\psi\}$ is the nodal

displacement vector of the entire column (buckling mode shape): $Det([K] - \lambda [K_p]) = 0$ (3.3)

$$[K]_e = \frac{E_0 I}{L_e^3} \begin{bmatrix} 12 & 6L_e & -12 & 6L_e \\ & 4L_e^2 & -6L_e & 2L_e^2 \\ & & 12 & -6L_e \\ \text{sym.} & & & 4L_e^2 \end{bmatrix} \quad (3.4); \quad [K_p]_e = \begin{bmatrix} \frac{6}{5L_e} & \frac{1}{10} & -\frac{6}{5L_e} & \frac{1}{10} \\ & \frac{2L_e}{15} & -\frac{1}{10} & -\frac{L_e}{30} \\ & & \frac{6}{5L_e} & -\frac{1}{10} \\ \text{sym.} & & & \frac{2L_e}{15} \end{bmatrix} \quad (3.5)$$

3.2. Development of the Stochastic Finite Element Method for Euler Buckling Analysis

Consider a column with a rectangular cross-section whose elastic modulus varies as a spatial random field (2D and 3D). The stability equilibrium differential equation of the column is given by Equation (3.6).

$$\frac{d^2}{dx^2} \left[\mathcal{D}(x, \theta) \frac{d^2 w(x, \theta)}{dx^2} \right] + P \frac{d^2 w(x, \theta)}{dx^2} = 0 \quad (3.6)$$

$$\text{With the effective bending stiffness determined as Equation (3.7): } \mathcal{D}(x, \theta) = \int_A y_e^2 E(\mathbf{x}, \theta) dydz \quad (3.7)$$

The effective stiffness matrix of the beam-column element is given by Equation (3.8) for the two-dimensional random elastic modulus case, and by Equation (3.9) for the three-dimensional case.

$$[k]_e = \underbrace{\int_{V_e} y_e^2 [N]^T E_0 [N] dV_e}_{[K_0]_e} + \underbrace{\int_{V_e} y_e^2 [N]^T E_0 f_e(x_e, y_e) [N] dV_e}_{\sum_{i=1}^{N_w} [\Delta K_i]_e \Theta_{i,e}} - [K_P]_e \quad (3.8)$$

$$[k]_e = \underbrace{\int_{V_e} y_e^2 [N]^T E_0 [N] dV_e}_{[K_0]_e} + \underbrace{\int_{V_e} y_e^2 [N]^T E_0 f_e(x_e, y_e, z_e) [N] dV_e}_{\sum_{i=1}^{N_w} [\Delta K_i]_e \Theta_{i,e}} - [K_P]_e \quad (3.9)$$

The basic random variables discretized from the random field are given in Equations (3.10), (3.11).

$$\Theta_{i,e} = \int_{A_e} y_e^2 x_e^{i-1} f_e(x_e, y_e) dA_e \quad (3.10); \quad \Theta_{i,e} = \int_{V_e} y_e^2 x_e^{i-1} f_e(x_e, y_e, z_e) dV_e \quad (3.11)$$

The global stiffness matrix $[k]$ of the entire column structure is assembled from $[k_e]$ as given in Equation (3.12).

$$[k] = \sum_{e=1}^{N_e} [k_e] = \underbrace{\sum_{e=1}^{N_e} [K_0]_e^{Ext}}_{[K(\{\lambda\})]} + \underbrace{\sum_{e=1}^{N_e} \sum_{i=1}^{N_w} [\Delta K_i]_e^{Ext} \Theta_{i,e}}_{[K_P]} - P \underbrace{\sum_{e=1}^{N_e} [K_P]_e^{Ext}}_{[K_P]} \quad (3.12)$$

For the k -th buckling mode, the buckling equation is written as (3.13):

$$([K(\{\Theta\})] - \lambda^k(\{\Theta\})[K_P])\{\psi^k(\{\Theta\})\} = \{0\} \quad (3.13)$$

Expanding the quantities appearing in Equation (3.13) in a Taylor series around their expected values yields Equations (3.14) - (3.16).

$$[K] \approx \sum_{e=1}^{N_e} [K_0]_e^{Ext} + \sum_{e=1}^{N_e} \sum_{i=1}^{N_w} \left[\frac{\partial [K(\{\Theta\})]}{\partial \Theta_{i,e}} \right]_{\bar{\Theta}_{i,e}} \Theta_{i,e} + \frac{1}{2} \sum_{e_1=1}^{N_e} \sum_{e_2=1}^{N_e} \sum_{i_1=1}^{N_w} \sum_{i_2=1}^{N_w} \left[\frac{\partial^2 [K(\{\Theta\})]}{\partial \Theta_{i_1,e_1} \partial \Theta_{i_2,e_2}} \right]_{\bar{\Theta}_{i_1,e_1}} \Theta_{i_1,e_1} \Theta_{i_2,e_2} + \dots \quad (3.14)$$

$$\lambda^k(\{\Theta\}) \approx \lambda_0^k + \sum_{e=1}^{N_e} \sum_{i=1}^{N_w} \left[\frac{\partial \lambda^k(\{\Theta\})}{\partial \Theta_{i,e}} \right]_{\bar{\Theta}_{i,e}} \Theta_{i,e} + \frac{1}{2} \sum_{e_1=1}^{N_e} \sum_{e_2=1}^{N_e} \sum_{i_1=1}^{N_w} \sum_{i_2=1}^{N_w} \left[\frac{\partial^2 \lambda^k(\{\Theta\})}{\partial \Theta_{i_1,e_1} \partial \Theta_{i_2,e_2}} \right]_{\bar{\Theta}_{i_1,e_1}} \Theta_{i_1,e_1} \Theta_{i_2,e_2} + \dots \quad (3.15)$$

$$\{\psi^k\} = \{\psi_0^k\} + \sum_{e=1}^{N_e} \sum_{i=1}^{N_w} \left[\frac{\partial \{\psi^k(\{\Theta\})\}}{\partial \Theta_{i,e}} \right]_{\bar{\Theta}_{i,e}} \Theta_{i,e} + \frac{1}{2} \sum_{e_1=1}^{N_e} \sum_{e_2=1}^{N_e} \sum_{i_1=1}^{N_w} \sum_{i_2=1}^{N_w} \left[\frac{\partial^2 \{\psi^k(\{\Theta\})\}}{\partial \Theta_{i_1,e_1} \partial \Theta_{i_2,e_2}} \right]_{\bar{\Theta}_{i_1,e_1}} \Theta_{i_1,e_1} \Theta_{i_2,e_2} + \dots \quad (3.16)$$

The first-order partial derivative of the k -th critical load eigenvalue with respect to $\Theta_{i,e}$ is given by (3.17):

$$\left[\frac{\partial \lambda^k(\{\Theta\})}{\partial \Theta_{i,e}} \right]_{\bar{\Theta}_{i,e}} = \frac{\{\psi_o^k\}^T \left[\frac{\partial [K(\{\Theta\})]}{\partial \Theta_{i,e}} \right]_{\bar{\Theta}_{i,e}} \{\psi_o^k\}}{\{\psi_o^k\}^T [K_P] \{\psi_o^k\}} \quad (3.17)$$

The first-order approximation of the k -th critical load with respect to the random variables is given by Equation (3.18).

$$\lambda^k(\{\Theta\}) \approx \lambda_0^k + \sum_{e=1}^{N_e} \sum_{i=1}^{N_w} \frac{\{\psi_o^k\}^T \left[\frac{\partial [K(\{\Theta\})]}{\partial \Theta_{i,e}} \right]_{\bar{\Theta}_{i,e}} \{\psi_o^k\}}{\{\psi_o^k\}^T [K_P] \{\psi_o^k\}} \Theta_{i,e} \quad (3.18)$$

The expectation and variance of the k -th critical load eigenvalue are given by Equation (3.19), **Error! Reference source not found.**

$$\mathbb{E}(\lambda^k) = \mathbb{E} \left[\lambda_0^k + \sum_{e=1}^{N_e} \sum_{i=1}^{N_w} \frac{\left\{ \psi_o^k \right\}^T \left[\frac{\partial [K(\{\Theta\})]}{\partial \Theta_{i,e}} \right]_{\bar{\Theta}_{i,e}} \left\{ \psi_o^k \right\}}{\left\{ \psi_o^k \right\}^T [K_P] \left\{ \psi_o^k \right\}} \Theta_{i,e} \right] \simeq \lambda_0^k \quad (3.19)$$

$$Var(\lambda^k) = \sum_{e_1=1}^{N_e} \sum_{e_2=1}^{N_e} \sum_{i_1=1}^{N_w} \sum_{i_2=1}^{N_w} \left\{ \frac{\left\{ \psi_o^k \right\}^T \left[\frac{\partial [K(\{\Theta\})]}{\partial \Theta_{i_1,e_1}} \right]_{\bar{\Theta}_{i_1,e_1}} \left\{ \psi_o^k \right\} \left\{ \psi_o^k \right\}^T \left[\frac{\partial [K(\{\Theta\})]}{\partial \Theta_{i_2,e_2}} \right]_{\bar{\Theta}_{i_2,e_2}} \left\{ \psi_o^k \right\}}{\left\{ \psi_o^k \right\}^T [K_P] \left\{ \psi_o^k \right\} \left\{ \psi_o^k \right\}^T [K_P] \left\{ \psi_o^k \right\}} \mathbb{E} \left[\Theta_{i_1,e_1} \Theta_{i_2,e_2} \right] \right\} \quad (3.20)$$

The variance of the critical load of the column with a two-dimensional random elastic modulus (3.21).

$$Var(\lambda^k) = \sum_{e_1=1}^{N_e} \sum_{e_2=1}^{N_e} \sum_{i_1=1}^{N_w} \sum_{i_2=1}^{N_w} \left\{ \frac{\left\{ \psi_o^k \right\}^T \left[\frac{\partial [K(\{\Theta\})]}{\partial \Theta_{i_1,e_1}} \right]_{\bar{\Theta}_{i_1,e_1}} \left\{ \psi_o^k \right\} \left\{ \psi_o^k \right\}^T \left[\frac{\partial [K(\{\Theta\})]}{\partial \Theta_{i_2,e_2}} \right]_{\bar{\Theta}_{i_2,e_2}} \left\{ \psi_o^k \right\}}{\left\{ \psi_o^k \right\}^T [K_P] \left\{ \psi_o^k \right\} \left\{ \psi_o^k \right\}^T [K_P] \left\{ \psi_o^k \right\}} \int_{A_{e_1}} \int_{A_{e_2}} y_{e_1}^2 x_{e_1}^{i_1-1} y_{e_2}^2 x_{e_2}^{i_2-1} R_{f_{e_1} f_{e_2}} dA_{e_1} dA_{e_2} \right\} \quad (3.21)$$

The variance of the critical load of the column with a three-dimensional random elastic modulus (3.22).

$$Var(\lambda^k) = \sum_{e_1=1}^{N_e} \sum_{e_2=1}^{N_e} \sum_{i_1=1}^{N_w} \sum_{i_2=1}^{N_w} \left\{ \frac{\left\{ \psi_o^k \right\}^T \left[\frac{\partial [K(\{\Theta\})]}{\partial \Theta_{i_1,e_1}} \right]_{\bar{\Theta}_{i_1,e_1}} \left\{ \psi_o^k \right\} \left\{ \psi_o^k \right\}^T \left[\frac{\partial [K(\{\Theta\})]}{\partial \Theta_{i_2,e_2}} \right]_{\bar{\Theta}_{i_2,e_2}} \left\{ \psi_o^k \right\}}{\left\{ \psi_o^k \right\}^T [K_P] \left\{ \psi_o^k \right\} \left\{ \psi_o^k \right\}^T [K_P] \left\{ \psi_o^k \right\}} \int_{V_{e_1}} \int_{V_{e_2}} y_{e_1}^2 x_{e_1}^{i_1-1} y_{e_2}^2 x_{e_2}^{i_2-1} R_{f_{e_1} f_{e_2}} dV_{e_1} dV_{e_2} \right\} \quad (3.22)$$

The coefficient of variation (COV) of the critical load is given by Equation (3.23).

$$COV = \frac{\sqrt{Var[\lambda^k]}}{\mathbb{E}[\lambda^k]} \quad (3.23)$$

3.3. Stochastic Euler Buckling Analysis of Columns with Spatially Random Material Properties Using MCS

Using the spectral representation method, the two-dimensional random elastic modulus field is generated as given in Equation (3.24), and the three-dimensional random field is presented in Equation (3.25).

$$f(x, y) = \sqrt{2} \sum_{n_1=0}^{N_1-1} \sum_{n_2=0}^{N_2-1} \left\{ A_{n_1 n_2} \cos \left[\kappa_{1n_1} x + \kappa_{2n_2} y + \Phi_{n_1 n_2}^{(1)} \right] + B_{n_1 n_2} \cos \left[\kappa_{1n_1} x - \kappa_{2n_2} y + \Phi_{n_1 n_2}^{(2)} \right] \right\} \quad (3.24)$$

$$f(x, y, z) = \sqrt{2} \sum_{n_1=0}^{N_1-1} \sum_{n_2=0}^{N_2-1} \sum_{n_3=0}^{N_3-1} \left\{ A_{n_1 n_2 n_3}^{(1)} \cos \left[\kappa_{1n_1} x + \kappa_{2n_2} y + \kappa_{3n_3} z + \Phi_{n_1 n_2 n_3}^{(1)} \right] + A_{n_1 n_2 n_3}^{(2)} \cos \left[\kappa_{1n_1} x + \kappa_{2n_2} y - \kappa_{3n_3} z + \Phi_{n_1 n_2 n_3}^{(2)} \right] \right. \\ \left. + A_{n_1 n_2 n_3}^{(3)} \cos \left[\kappa_{1n_1} x - \kappa_{2n_2} y + \kappa_{3n_3} z + \Phi_{n_1 n_2 n_3}^{(3)} \right] + A_{n_1 n_2 n_3}^{(4)} \cos \left[\kappa_{1n_1} x - \kappa_{2n_2} y - \kappa_{3n_3} z + \Phi_{n_1 n_2 n_3}^{(4)} \right] \right\} \quad (3.25)$$

Based on the generated dataset combined with the standard FEM, the k -th critical load eigenvalue in the i -th simulation can be determined as shown in the following Equation (3.26).

$$Det \left(\sum_{e=1}^{N_e} [K_0]_e^{Ext} + \sum_{e=1}^{N_e} [\Delta K]_e^{Ext} - \lambda_i^k \sum_{e=1}^{N_e} [K_P]_e^{Ext} \right) = 0 \quad (3.26)$$

Based on Equation (3.26), N values of the k -th critical load eigenvalue are obtained. Through statistical processing of the obtained dataset, the sample statistical characteristics, including the mean and variance, can be determined using the following equations:

$$\bar{\lambda}_k = \frac{1}{N} \sum_{i=1}^N \lambda_i^k \quad (3.27); \quad s_{\lambda_k}^2 = \frac{1}{N-1} \sum_{i=1}^N \left(\lambda_i^k - \bar{\lambda}_k \right)^2 \quad (3.28)$$

$$\text{The COV of the critical load eigenvalue is given by Equation(3.29): } COV = \frac{\sqrt{s_{\lambda_k}^2}}{\bar{\lambda}_k} \quad (3.29)$$

3.4. Validation of the stochastic finite element model for Euler buckling analysis of columns with spatially random material properties

3.4.1. Comparison between plane and spatial beam models in stochastic buckling analysis

To comprehensively analyze the stability of the column under the effect of a three-dimensional random elastic modulus field, the system is initially modeled using a spatial beam element (12 degrees of freedom). Mechanically, material fluctuations across the cross-section generate random characteristics, including first moments of area (S_y , S_z) and product of inertia stiffness (D_{yz}), which directly induce spatial coupling effects in the stiffness matrix. However, from a statistical perspective, since the material field varies around a constant expected value, the theoretical expectations of these eccentric quantities are zero. The inevitable mathematical consequence is that, in the mean state, the coupling matrix blocks gradually vanish, causing the 3D spatial problem to degenerate and decouple into independent plane stability problems.

To verify this, the dissertation investigates the Euler critical load on a centrally compressed cantilever column (fixed-free, $L = 8m$, cross-section $b = 0.25m$, $h = 0.2m$, expected value $E_0 = 30 \times 10^9 \text{ N/m}^2$, and a coefficient of variation of 20%). The statistical results confirm that the discrepancies in the expected value and standard deviation of the critical load between the 3D model and the plane beam model are marginal (a maximum of only 1.33% for the expected value and 2.67% for the standard deviation). This numerical convergence provides solid evidence to conclude that applying a simplified plane beam model in 3D stochastic stability analysis is reasonable while effectively optimizing computational effort.

3.4.2. Convergence Study of the Critical Load Coefficient of Variation with Respect to Element Mesh Density

The problem investigates a centrally compressed column member (length $8m$, cross-section $0.25m \times 0.25m$) under four boundary conditions: fixed-fixed (C-C), fixed-pinned (C-H), pinned-pinned (H-H), and fixed-free (C-F). The elastic modulus is modeled as a stationary, spatially univariate Gaussian random field (2D and 3D), with an expected value $E_0 = 30 \times 10^9 \text{ N/m}^2$, standard deviation $\sigma_f = 0.05$ and a correlation length varying within $[0.01L, 100L]$.

To verify numerical stability, the COV of the Euler critical load is analyzed through discretization schemes with the number of elements $N_e = \{20, 30, 40\}$. The investigation results indicate that the statistical response of the critical load converges when the mesh density reaches 40 elements. The numerical solution maintains high stability, independent of the mesh density. Based on this foundation, the mesh of $N_e = 40$ is selected for all subsequent analysis steps, ensuring an optimal balance between the accuracy of the results and computational efficiency.

3.4.3. Verification of the Stochastic Finite Element Model Based on Stochastic Computational Theory

According to stochastic mechanics, as the fluctuation of the material property field approaches zero, the non-deterministic system converges to a deterministic state. To validate this theoretical characteristic, the dissertation investigates the model with an elastic modulus field having an extremely small standard deviation ($\sigma_f = 10^{-5}$). The analysis results demonstrate that the COV of the Euler critical load simultaneously decreases and approaches a negligibly small value. The complete elimination of the random component in the column's response accurately reflects the physical asymptotic behavior of the system. This mathematical convergence provides foundational evidence to affirm the correctness, numerical stability, and reliability of the SFEM theoretical framework developed for stochastic stability analysis.

3.4.4. Verification of the Stochastic Finite Element Model against Published Research

To validate the reliability of the SFEM theoretical framework, the stochastic stability problem is directly compared with Reference [136] on a centrally compressed pinned-pinned column member ($L = 2m$, cross-section $0.3m \times 0.3m$, $E_0 = 30 \times 10^9 \text{ N/m}^2$, Reference [136] employs the orthogonal series expansion method and second-order perturbation to solve the one-dimensional (1D) random stiffness field problem. The dissertation sets the axial correlation length d_x identically to [136] ($d_x = 4m$ and $12m$), while simultaneously amplifying the transverse directions ($d_y = d_z = 100L$) to eliminate cross-sectional fluctuations, forcing the 2D/3D random field to perfectly approximate 1D behavior.

The expected value of the normalized critical load from the proposed SFEM model converges to the theoretical value, with a minimal error compared to [136] (consistently under 1%). Similarly, the coefficient of variation (COV) also records a strictly controlled deviation below the 5% threshold. This discrepancy can be entirely explained by the inherent differences in the random field discretization techniques and the perturbation expansion order between the two studies. This verification result not only affirms the reliability of the proposed SFEM algorithm but also demonstrates the comprehensiveness of the multi-dimensional model: a generalized 3D model is fully capable of degenerating to accurately solve traditional 1D problems.

3.4.5. Comparison of the Stochastic Finite Element Euler Buckling Analysis with Monte Carlo Simulation

To independently verify the reliability of the SFEM algorithm in the stability problem, the dissertation implements a Monte Carlo Simulation (MCS) based on the spectral representation technique combined with the standard FEM, utilizing a sample size of $N = 10,000$ realizations of the spatial random elastic modulus field. Statistical evaluation of the unnormalized critical load eigenvalue responses indicates an approximation to the Gaussian distribution, evidenced by the Q-Q plot closely following the theoretical reference line, along with skewness and kurtosis coefficients approaching standard values. Quantitative comparison of the expectation and COV between the two methods records minor differences. Notably, in regions of short spatial correlation (where material

fluctuations are intense), the COV from the MCS tends to be larger than that of the SFEM solution; however, this deviation rapidly decreases as the correlation length increases, accurately reflecting the uniform asymptotic state of the material field. With an error margin below the 5% threshold, these numerical experimental results firmly affirm the accuracy and suitability of the proposed SFEM theoretical framework.

3.5. Conclusions of Chapter III

Building upon the foundation of the weighted integral technique established in Chapter II, the SFEM theoretical framework has been successfully expanded and formulated in Chapter III to comprehensively solve the Euler buckling problem of centrally compressed columns under the influence of spatially random elastic modulus fields (2D and 3D). The core scientific results and contributions include:

- **On the development of the computational theoretical framework:** The dissertation has successfully integrated the perturbation technique to establish a first-order Taylor series approximation for the critical load. This SFEM framework allows for the direct propagation of spatial uncertainty from the material structure, thereby explicitly determining the low-order statistical characteristics (expectation, variance, and coefficient of variation) of the Euler critical load without requiring iterative computational costs.
- **On the verification of reliability and accuracy:** The proposed SFEM algorithm has demonstrated its accuracy and mathematical stability through three rigorous criteria:
 - ✓ Absolute numerical convergence, which is completely independent of the element mesh density.
 - ✓ Perfect asymptotic convergence to the solution of the deterministic problem as material fluctuations approach zero, ensuring consistency with the foundational principles of stochastic mechanics.
 - ✓ Attaining consistent accuracy (with the error margin strictly controlled below 5%) when cross-validated with published research and statistical methods based on Monte Carlo Simulation (MCS).

CHAPTER IV: INVESTIGATION OF THE INFLUENCE OF RANDOM FIELD PARAMETERS ON THE STATISTICAL RESPONSE CHARACTERISTICS OF BEAMS AND COLUMNS

4.1. Quantitative investigation of the stochastic static response of beams with spatially random elastic modulus

4.1.1. Analysis of the influence of correlation length on the static behavior of beams with random material properties

4.1.1.1. Two-dimensional random elastic modulus

To evaluate the influence of the spatial correlation structure on the dispersion of the structural response, the dissertation investigates the variation of the displacement coefficient of variation (COV) with respect to the correlation lengths d_x and d_y in the range from $0.01L$ to $100L$ (**Figure 4.1**). Graphical analysis shows that the displacement COV increases as the correlation length increases. Specifically, in the region of short correlation lengths, the COV increases rapidly (almost linearly) because the material field fluctuates intensely; consequently, the spatial averaging effect is maximized, thereby reducing the dispersion of the global response. Conversely, when the correlation length exceeds the characteristic dimension of the structure, the growth rate of the COV gradually saturates. At the theoretical limit where the correlation length approaches infinity equivalent to the random field degenerating into a uniform random variable over the entire domain the displacement COV asymptotes to the standard deviation of the input material field. This variation pattern reflects the sensitivity of the structure to spatial randomness and strictly complies with the fundamental principles of stochastic mechanics.

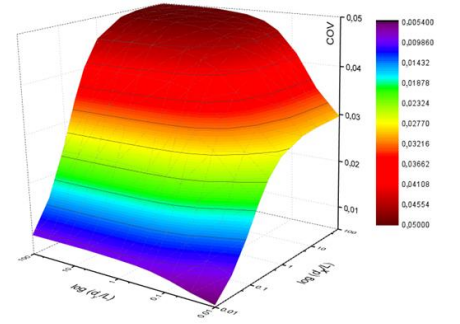
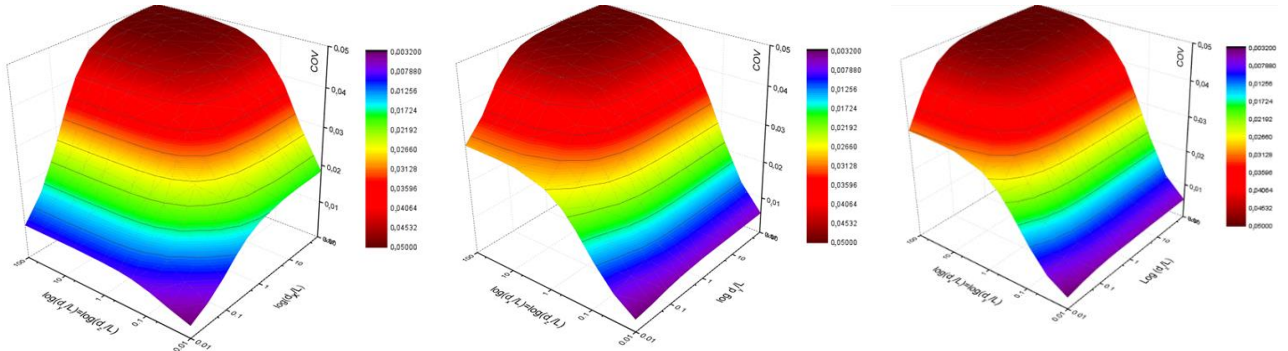


Figure 4.1. Influence of correlation length in two-dimensional space on the displacement COV

4.1.1.2. Three-dimensional random elastic modulus



a, $d_x = d, d_y = d_z = d$

b, $d_y = d, d_x = d_z = d$

c, $d_z = d, d_x = d_y = d$

Figure 4.2. Influence of correlation length in three-dimensional space on the displacement COV

Expanding the analysis to the three-dimensional random field (**Figure 4.2**), the dissertation evaluates the degree of influence of the correlation length components (d_x, d_y, d_z) on the COV of the mid-span displacement within the investigated range from $0.01L$ to $100L$. The results reflect a consistent pattern across all scenarios. In the short correlation range, the COV exhibits very high sensitivity to the increase in correlation length; typically, in the scenario where $d_x = d_y = d_z = d$, as d increases by a factor of 2.67 (from $0.03L$ to $0.08L$), the COV sharply increases by a factor of 2.15 (from 0.0086 to 0.0185). Conversely, when exceeding the characteristic dimension of the structure and entering the large correlation range, the increase saturates; as evidenced by the fact that when d continues to increase by a factor of 2.5 (from $40L$ to $100L$), the COV response remains almost constant (with only a marginal increase from 0.0498 to 0.0499). This nonlinear phenomenon affirms the mathematical asymptotic limit of the system: as d approaches infinity, the spatial random field asymptotes to a uniform state (degenerating into a global random variable), causing the displacement COV to absolutely converge to the exact initial standard deviation value of the material.

4.1.2. Analysis of the influence of the elastic modulus coefficient of variation on the displacement COV

4.1.2.1. Two-dimensional random elastic modulus

In the Gaussian random field setting, the coefficient of variation of the elastic modulus is mathematically equivalent to the input standard deviation σ_f . To evaluate the sensitivity of the system to the degree of material dispersion, the dissertation investigates the variation of the displacement COV with respect to σ_f in $[0.00, 0.20]$, combined with isotropic correlation lengths $d_x = d_y = d$ ranging from $0.01L$ to $100L$. The analysis results (**Figure 4.3**) confirm a consistent positive linear correlation across the entire investigated domain: the displacement COV always increases proportionally with the dispersion of the elastic modulus. Notably, the rate of increase of the output response is directly governed by the spatial structure of the material field. Specifically, the larger the correlation length d , the weaker the spatial averaging effect becomes, thereby strongly amplifying the impact of the fluctuation level σ_f on the global system.

4.1.2.2. Three-dimensional random elastic modulus

Expanding the investigation to the three-dimensional random field model, the study evaluates the sensitivity of the system through four levels of material fluctuation ($\sigma_f = 0.05, 0.10, 0.15, 0.20$) spanning the entire correlation length range from $0.01L$ to $100L$ (**Figure 4.4**). The analytical data reaffirm the consistency of the system with the mechanical behavior recorded in the 2D random model: the displacement COV always exhibits a positive, directly proportional correlation with the standard deviation of the elastic modulus.

4.1.3. Analysis of the relationship between the displacement coefficient of variation and the coefficient of variation of the spatially random elastic modulus

To quantitatively analyze the uncertainty propagation mechanism, **Figure 4.5** illustrates the relationship between the displacement COV and the standard deviation of the material field at four representative correlation lengths: $d = \{0.03L, 0.2L, 1L, 15L\}$ for both the 2D ($d_x = d_y = d$) and 3D ($d_x = d_y = d_z = d$) random models. The investigation results consistently confirm a single pattern across all scenarios: the dispersion level of the displacement response increases linearly and proportionally with the fluctuation of the elastic modulus. This linear dependence reflects the stable statistical propagation nature of the system, providing a theoretical basis for rapidly predicting the stochastic response characteristics of the beam given the input random.

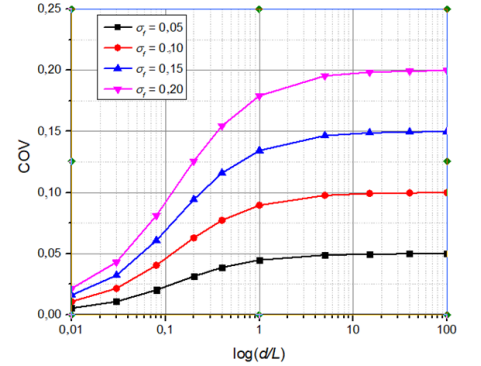


Figure 4.3. Influence of σ_f of the two-dimensional random field on the displacement COV

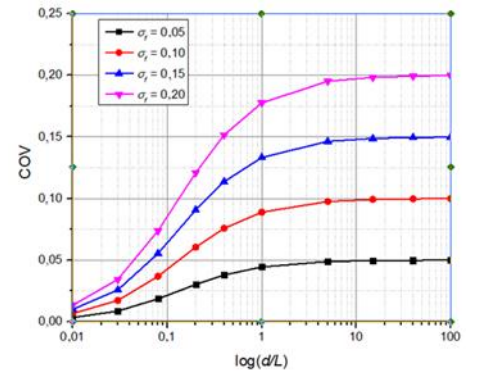


Figure 4.4. Effect of the standard deviation of the 3D random field on the coefficient of variation of displacement

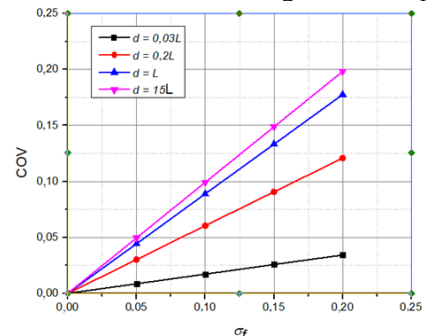
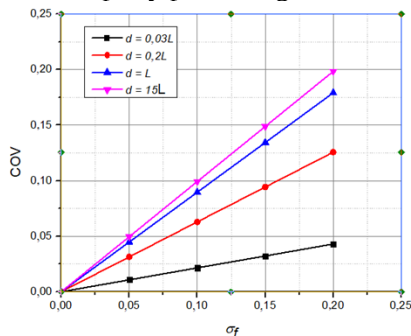
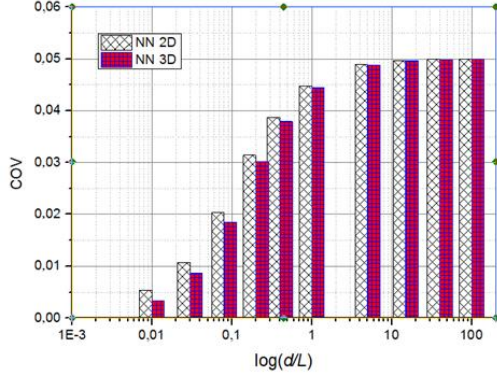


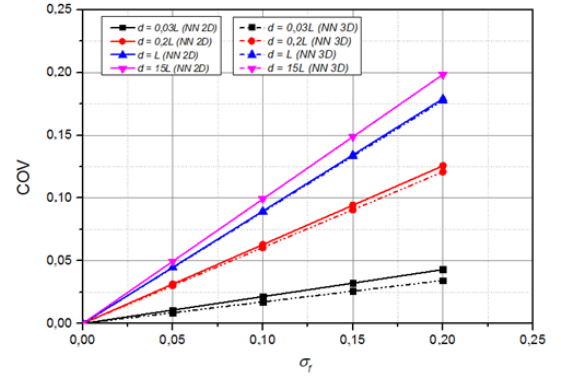
Figure 4.5. Relationship between the displacement COV and the COV of the random elastic modulus
a, Two-dimensional random elastic modulus b, Three-dimensional random elastic modulus

4.1.4. Comparison of the stochastic static response of beams with 2D and 3D random elastic modulus

Figure 4.6 directly compares the displacement dispersion of the beam under the influence of 2D and 3D random elastic modulus fields, with the correlation length varying within $[0.01L, 100L]$. In the short correlation region, intense fluctuations cause the material to fragment into numerous independent random volumes. Since the 3D field possesses a greater number of independent volumes compared to the 2D field, the spatial averaging effect is more pronounced. Consequently, the displacement COV in the 3D model decreases and remains smaller than that in the 2D model. Conversely, when the correlation length exceeds the characteristic dimension of the structure, both material fields approach a uniform state and simultaneously degenerate into a global random variable. At this limit, the statistical discrepancy between the 2D and 3D models is almost completely eliminated, demonstrating the convergence of the stochastic response regardless of the difference in spatial dimensionality.



a, Influence of the random field correlation length on the COV



b, Relationship between σ_f of the 2D and 3D random fields and the COV

Figure 4.6. Comparison of the static response of the beam with two- and three-dimensional random elastic modulus

4.2. Quantitative investigation of the linear dynamic characteristics of beams with spatially random material properties

4.2.1. Quantitative analysis of the linear dynamic behavior of beams with spatial random material properties

4.2.1.1. Two-dimensional random material properties

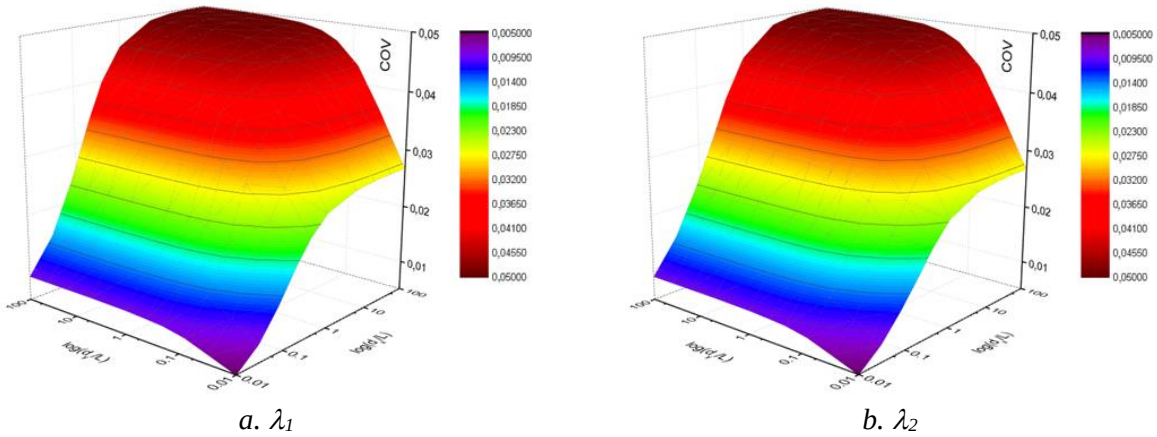


Figure 4.7. Influence of correlation length on the eigenvalue COV for beams with two-dimensional random material properties ($\gamma = -0.5$)

To evaluate the influence of the spatial structure on the linear dynamic characteristics, the study investigates the variation of the eigenvalue COV with respect to the 2D correlation length (in the range from $0.01L$ to $100L$) with an input dispersion level of $\sigma_f = \sigma_g = 0.05$. The investigation is comprehensively conducted across three independent scenarios: (i) only random E , (ii) only random ρ , and (iii) a random field including both E and ρ coupled via a cross-correlation coefficient $\gamma = \{-0.5, 0, 0.5\}$. The analytical data series (**Figure 4.7**) confirms a consistent pattern: in the short correlation range, the COV increases sharply due to the spatial averaging effect. However, as the correlation length exceeds the beam dimension and approaches infinity, the growth rate of the COV gradually saturates. This asymptote is mathematical evidence of the system's phase transition, where the material field completely converges to a uniform state, causing the beam's stochastic response to degenerate and become equivalent to a global random variable model.

4.2.1.2. Three-dimensional random material properties

To delve deeper into the uncertainty propagation mechanism in three-dimensional space, the dissertation expands the investigation of the eigenvalue COV under combinations of correlation lengths ($d_x = d_y, d_y = d_z, d_x = d_z$) for all three material distribution scenarios (random E , random ρ , and combined random E, ρ). The analytical data series affirms a general pattern: the increase in correlation length weakens the spatial averaging effect, leading to an increase in the dispersion level of the eigenvalue. However, the most significant finding recorded is the dominant

influence of the axial correlation length (d_x) compared to the cross-sectional directions (d_y, d_z). Mechanically, because the axis is the primary geometric and load-bearing direction of the beam, material fluctuations along the span length directly determine the distribution of "stiff-soft" regions, as well as the distribution of the mass participating in the global system's vibration. Consequently, the structural dynamics become extremely sensitive to the axial random component, affirming that the spatial directionality of the material field plays a decisive role in shaping the stochastic vibration characteristics.

4.2.2. Analysis of the influence of the material property coefficient of variation on the eigenvalue COV

4.2.2.1. Two-dimensional random material properties

The quantitative analysis clarifies the dominant influence of material property dispersion on the eigenvalue COV. The investigation results reveal a consistent pattern: the COV of the eigenvibration response always increases proportionally with the standard deviation of the input field. Specifically, at the limit of very large correlation lengths (when the random field degenerates into a global random variable), the dynamic characteristics exhibit strict mathematical asymptotic laws that depend on the cross-correlation coefficient between the elastic modulus and the mass density. In detail, when considering only a univariate field, the eigenvalue COV converges to the input standard deviation. Under perfectly negative correlation, the perturbation resonance effect between the random components of the stiffness matrix and the mass matrix reaches its maximum, and the COV asymptotes to the sum of the standard deviations of the two fields. Conversely, under perfectly positive correlation conditions, the in-phase fluctuations of the stiffness matrix and the mass matrix mutually compensate and cancel each other out, significantly minimizing the variation of the vibration response. Notably, when the two material fields are statistically independent, the COV converges to approximately 1.4 times the standard deviation of the input random field.

4.2.2.2. Three-dimensional random material properties

The influence of the three-dimensional random field standard deviation and correlation length on the eigenvalue COV is investigated. The random fields of E and ρ are assumed to vary correspondingly with values of 0.05, 0.10, 0.15, and 0.20, under the condition $d_x = d_y = d_z = d$. The analytical results are presented in **Figure 4.8**. The eigenvalue exhibits a trend of increasing fluctuation around its expected value as the standard deviation of the input random field or the correlation length increases. When the correlation length reaches a very large value, the COV tends to asymptote to the standard deviation of the random field in cases where the beam has only 3D random E or only 3D random ρ . In the case of beams with 3D random E and ρ having a perfectly negative correlation, as the correlation lengths along the spatial axes approach infinity, the eigenvalue COV tends to asymptote to the sum of the standard deviations of the random fields. Meanwhile, when the random fields are statistically independent, the value to which the COV asymptotes is approximately 70% of the sum of the standard deviations of the random fields.

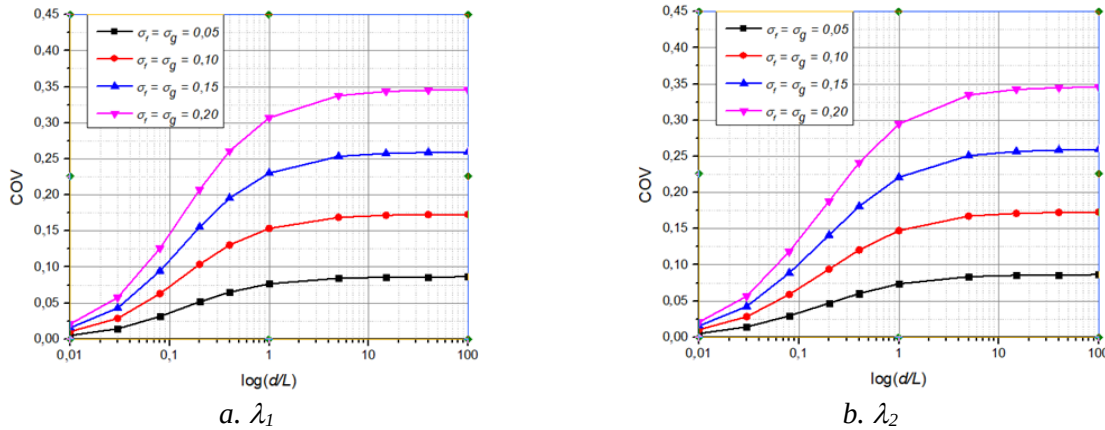


Figure 4.8. Influence of the 3D random field standard deviation on the eigenvalue COV ($\gamma = -0.5$)

4.2.3. Analysis of the relationship between the eigenvalue coefficient of variation and the coefficient of variation of spatially random material properties

4.2.3.1. Two-dimensional random material properties

The relationship between the eigenvalue COV and the standard deviation of the two-dimensional random field, which is essentially the material property COV. The standard deviation of the two-dimensional random field is investigated corresponding to the values: 0, 0.05, 0.10, 0.15, and 0.20. The correlation lengths along the axes are considered corresponding to four cases: $d_x = d_y = 0.03L$, $d_x = d_y = 0.2L$, $d_x = d_y = 1L$, $d_x = d_y = 15L$. In the case of beams with only 2D random E or ρ , the eigenvalue COV and the input material property COV exhibit a linear relationship. This indicates that the level of uncertainty propagation from input to output is linear when governed by only a single random source.

4.2.3.2. Three-dimensional random material properties

The relationship between the eigenvalue COV and the three-dimensional random field standard deviation is investigated for correlation length cases including: $d_x = d_y = d_z = 0.03L$, $d_x = d_y = d_z = 0.2L$, $d_x = d_y = d_z = 1L$, and $d_x = d_y = d_z = 15L$. Simultaneously, three 3D random cases of material properties are also studied in this section, including: beams with only 3D random elastic modulus, beams with only 3D random mass density, and beams with

both 3D random elastic modulus and mass density. The random fields considered are stationary random fields with a zero expected value. Therefore, the standard deviation of the random field can be considered approximately equal to the COV of the input random factor. When the beam has only 3D random E or ρ , a linear relationship between the input random factor COV and the eigenvalue COV is recorded. In the case of beams with simultaneously 3D random E and ρ , the relationship between the eigenvalue COV and the input random factor COV is approximately linear.

4.2.4. Analysis of the influence of the correlation between spatial random material property fields on the eigenvalue coefficient of variation

A quantitative investigation is conducted on the cross-correlation coefficient (γ) between the elastic modulus and mass density fields, with $\gamma = \{1.0, -0.5, 0.0, +0.5, +1.0\}$, combined with input fluctuation levels over the correlation length range from $0.01L$ to $100L$. The analytical results confirm a negative correlation: the eigenvalue COV decreases as γ increases. The response fluctuation reaches its maximum at the perfectly negative correlation threshold ($\gamma = -1.0$) and is almost completely extinguished at the perfectly positive correlation threshold ($\gamma = 1.0$). Particularly, at the asymptotic limit when the correlation length approaches infinity, the COV of the system with $\gamma = -1.0$ converges to twice the standard deviation value of the input field. The mechanical nature of this phenomenon is directly governed by the structure of the equation of motion. The stiffness matrix and mass matrix of the system are composed of both deterministic and random components. Under perfectly negative correlation, the out-of-phase variation creates a perturbation resonance effect, pushing the fluctuation level of the eigenvalue to its maximum. Conversely, under perfectly positive correlation, the in-phase variation of the two matrices creates a self-compensating mechanism, significantly canceling out the dispersion of the global response. This finding affirms that the cross-correlation coefficient is a conditional parameter in stochastic mechanics problems. Ignoring or falsely assuming the correlation characteristics between E and ρ will lead to errors in structural reliability assessment.

4.2.5. Comparison of the linear dynamic characteristics of beams with 2D and 3D random material properties

To evaluate the influence of spatial dimensionality on the uncertainty of linear dynamic characteristics, the study investigates the eigenvalue COV between the 2D and 3D random field models, as shown in **Figure 4.9**. The analytical data reaffirm the power of the spatial averaging mechanism: in short correlation regions, the increase in spatial dimensionality from 2D to 3D increases the number of independent random volumes. This increase reduces the overall dispersion level, causing the eigenvalue COV in the 3D model to decrease and remain consistently smaller than that of the 2D model. Conversely, when the correlation length is large, the material field in both models approaches a uniform state and simultaneously degenerates into a global random variable. At this asymptotic limit, the statistical discrepancy between the 2D and 3D models is almost zero, fully demonstrating the consistency of the system response regardless of the expansion in spatial dimensionality.

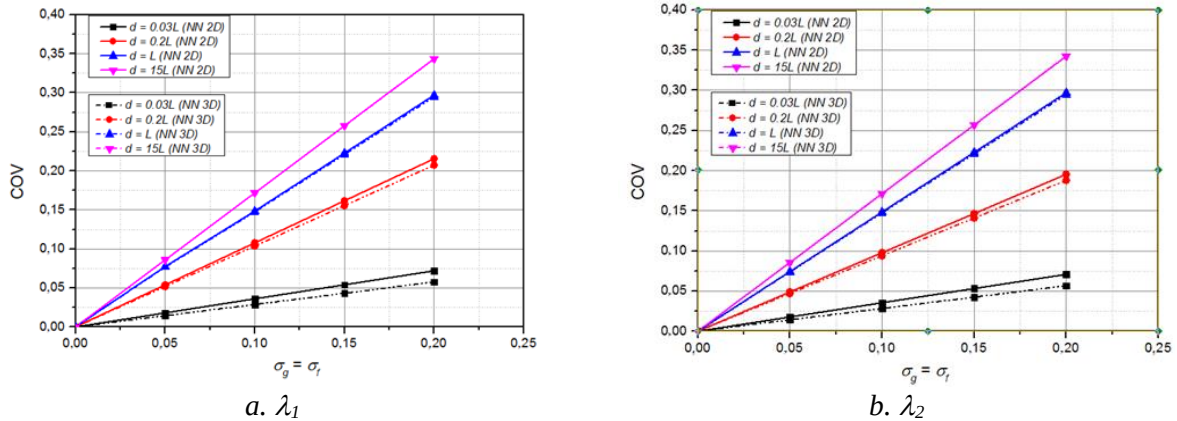


Figure 4.9. Comparison of the relationship between the eigenvalue COV and the COV of two- and three-dimensional random material property fields ($\gamma = -0.5$)

4.3. Quantitative investigation of the Euler buckling stability of columns with spatially random material properties

4.3.1. Effect of the correlation length of the random field on the coefficient of variation of the critical buckling load

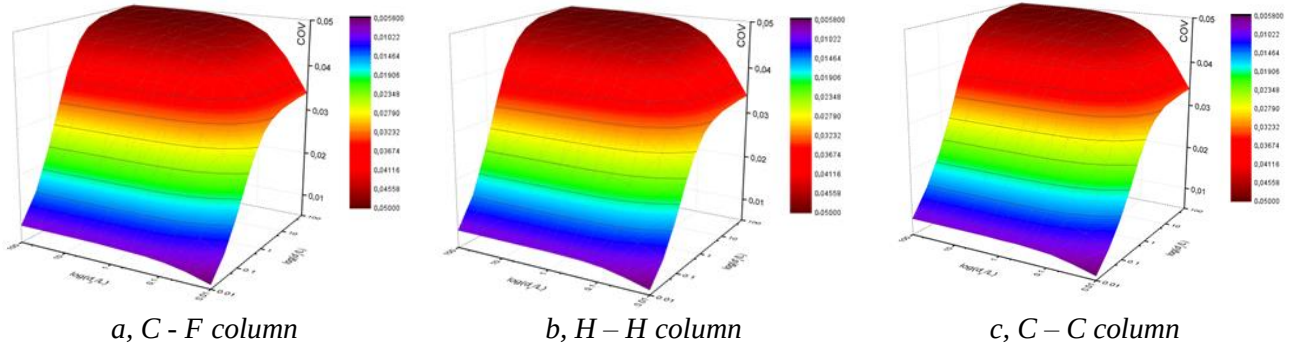
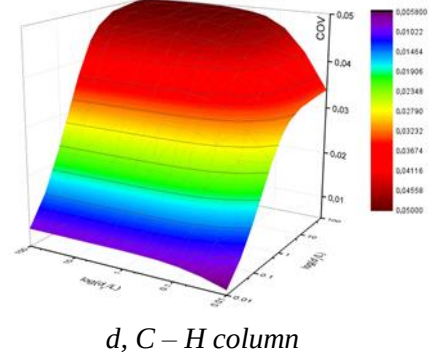


Figure 4.10. Influence of correlation length on the critical load COV for 2D random elastic modulus

To evaluate the impact of the spatial correlation structure on the stochastic stability of the column, the dissertation investigates the variation of the Euler critical load COV with respect to the correlation length ($0.01L$ to $100L$, with an input standard deviation of $\sigma_f = 0.05$) under various boundary conditions (**Figure 4.10**). The analytical data confirm a clear increasing trend: in the short correlation range, material fluctuations occur within a narrow scope, maximizing the spatial averaging effect. This mechanism effectively suppresses local perturbations, reducing the dispersion of the critical load. Conversely, as the correlation length increases, the spatial variation occurs more gradually, causing the spatial averaging effect to weaken and thereby elevating the fluctuation level of the entire system. At the asymptotic limit approaching infinity, the random field completely degenerates into a global random variable, causing the critical load COV to converge to the initial standard deviation of the elastic modulus. Furthermore, the investigation results reveal a specific spatial directionality: originating from its compressive nature, the column's stability state exhibits a pronounced sensitivity to the axial perturbation component compared to the transverse correlation lengths.



4.3.2. Analysis of the influence of the elastic modulus coefficient of variation on the critical load COV

To quantify the dominant influence of the material property fluctuation amplitude on the structural stability state, the study expands the investigation of the critical load COV with respect to the random field standard deviation, combined over the correlation length range from $0.01L$ to $100L$. The analytical data for both the two-dimensional (**Figure 4.11**) and three-dimensional (**Figure 4.12**) random models affirm a strong positive correlation: the critical load COV increases proportionally with the standard deviation of the spatial field (the coefficient of variation of the elastic modulus). Mechanically, the amplification of the input standard deviation directly widens the dispersion of the elastic modulus around its expected value, inevitably leading to a corresponding increase in the dispersion of the critical load. Simultaneously, when referenced along the correlation length axis, the system response continues to exhibit non-linearity: the COV increases sharply in the short correlation range but gradually saturates in the large correlation range, perfectly matching the established theoretical asymptotic laws.

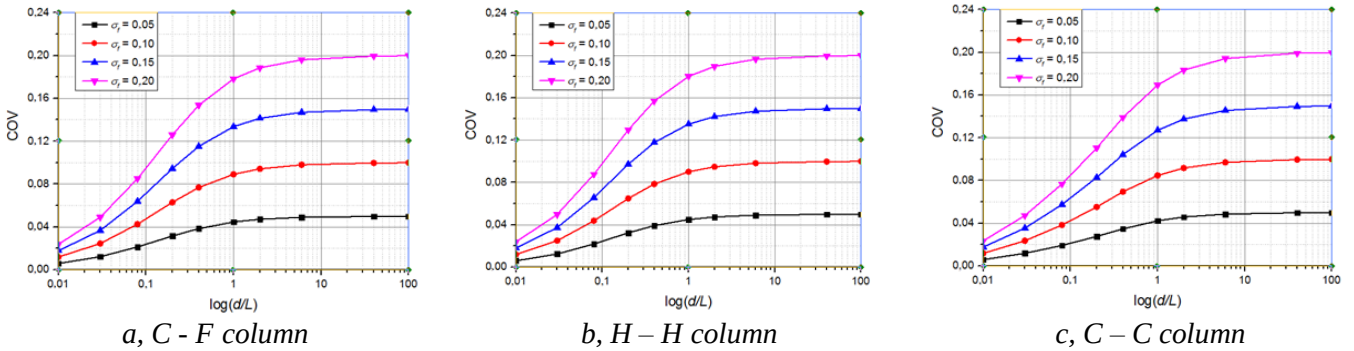


Figure 4.11. Influence of the 2D random field standard deviation on the critical load COV

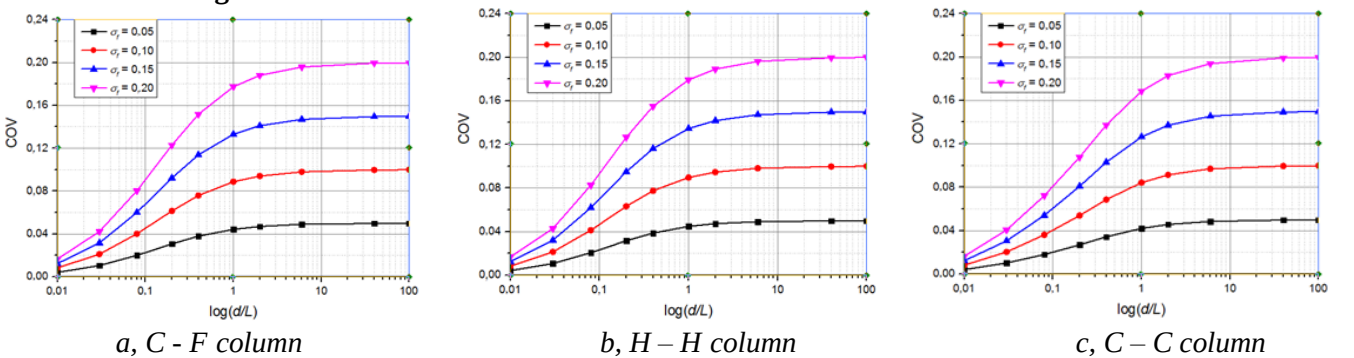


Figure 4.12. Influence of the 3D random field standard deviation on the critical load COV

4.3.3. Analysis of the relationship between the critical load coefficient of variation and the coefficient of variation of the spatially random elastic modulus

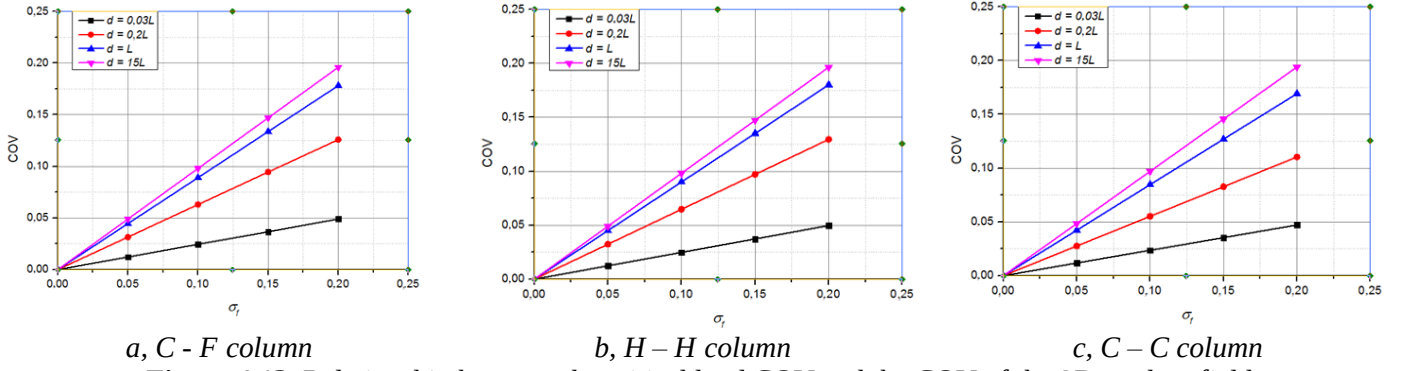


Figure 4.13. Relationship between the critical load COV and the COV of the 2D random field

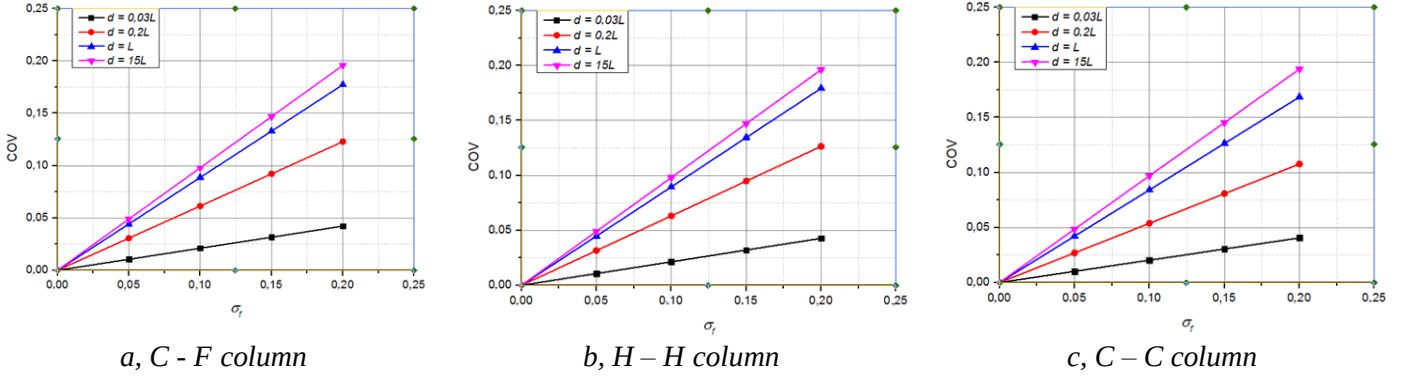
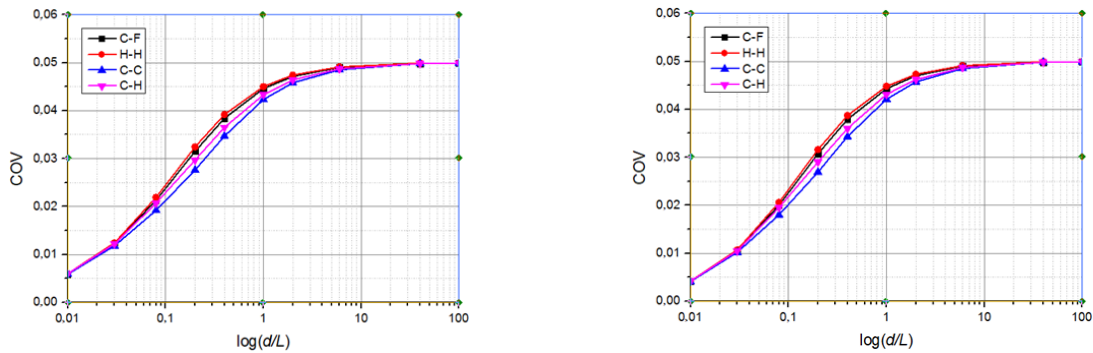


Figure 4.14. Relationship between the critical load COV and the COV of the 3D random field

To quantify the uncertainty propagation mechanism in the stability problem, the study investigates the relationship between the critical load COV and the dispersion level of the material field. The investigation is conducted synchronously on both the two-dimensional (**Figure 4.13**) and three-dimensional (**Figure 4.14**) random field models at: $d = \{0.03L, 0.2L, 1L, 15L\}$. The analytical data confirm a consistent pattern across all scenarios: there exists a direct linear correlation between the critical load COV and the elastic modulus coefficient of variation. This finding regarding linearity brings practical value to structural design, providing a basis for predicting the column's stochastic stability reserve solely through the basic statistical characteristics of the input material.

4.3.4. Influence of column boundary conditions on the critical load coefficient of variation

The boundary conditions at the column ends have a direct influence on the magnitude of the critical load and the buckling mode of the column. When the column features a spatially random elastic modulus, the critical load also becomes a random variable. The influence of the column boundary conditions on the COV is relatively small compared to the impact of material property randomness, especially at short correlation lengths or large correlation lengths approaching infinity, as illustrated in **Figure 4.15**.



a, $\sigma_f = 0.05$; two-dimensional random field

b, $\sigma_f = 0.05$; three-dimensional random field

Figure 4.15. Influence of boundary conditions on the critical load COV for spatially random elastic modulus

4.3.5. Comparison of Euler buckling of columns with two- and three-dimensional stochastic elastic moduli

When the material's elastic modulus is modeled as a two-dimensional random field, its spatial random variation is only considered within a two-dimensional plane, i.e., along the longitudinal axis and the height of the cross-section. Conversely, in the case of a three-dimensional random field, the elastic modulus is randomly distributed throughout the entire volume of the column, more comprehensively reflecting the spatial material heterogeneity. This leads to a difference in the stochastic characteristics of the response, as illustrated in **Figure 4.16**. Specifically, when the spatial correlation length is smaller than the characteristic dimension of the column, the critical load COV for the three-dimensional elastic modulus is smaller than that of the two-dimensional case. This indicates

that the stochastic response of the 3D column exhibits a smaller dispersion level around its expected value, reflecting higher statistical stability. Conversely, as the spatial correlation length increases and becomes large relative to the column dimensions, the critical load COV in both cases becomes nearly identical because the spatial random fields degenerate into global random variables.

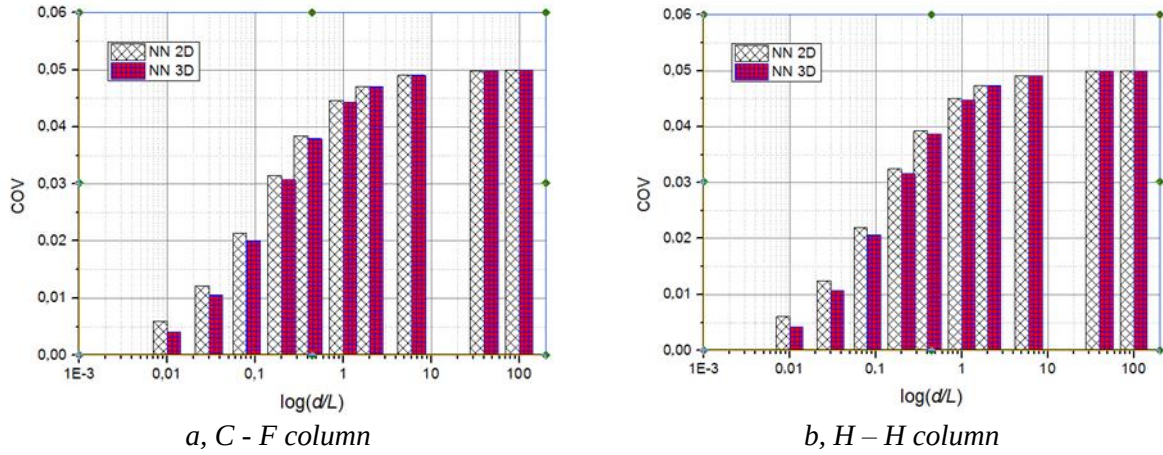


Figure 4.16. Comparison of the critical load COV for two- and three-dimensional random E

4.4. Chapter IV Conclusions

Based on the SFEM model established and verified in the previous chapters, the influence of the input spatially random material property fields on the mechanical behavior of beams and columns has been quantitatively investigated as the focal point of Chapter IV. From the analytical results, the core laws have been deduced, including:

Influence of correlation length on the stochastic response of beams and columns:

- ✓ **Statistical averaging effect:** At short correlation lengths, the rapid random variation of material properties induces a strong averaging effect, reducing the coefficient of variation (COV) of the stochastic response of beams and columns.
- ✓ **Dispersion saturation:** As the correlation length increases, the stochastic response COV increases rapidly and gradually saturates. When the correlation length approaches infinity, the random field transitions into a global random variable; at this state, the response COV of the beams and columns asymptotically approaches the input material property COV.
- ✓ **Directional dominance:** The axial correlation length plays a dominant role in the response dispersion compared to the other directions.

Influence of stochastic spatial dimensionality (2D, 3D):

- ✓ The comparative results reveal a distinct difference in the averaging effect between the 2D and 3D random models. Within the short correlation length range, the 3D random model exhibits a more robust averaging effect, leading to a smaller stochastic response COV compared to the 2D model.
- ✓ However, at extremely large correlation lengths, the discrepancy between these two models is eliminated as both converge to the uniform random variable problem.

Influence of cross-correlation: For the analysis of linear dynamic characteristics, the correlation between the random fields E and ρ strongly governs the dispersion of the eigenvalue:

- ✓ **Perfectly negative correlation:** Induces an additive dispersion phenomenon, causing the eigenvalue COV to reach its maximum (approaching the sum of the COVs of E and ρ as the correlation length tends to infinity).
- ✓ **Perfectly positive correlation:** Induces a stochastic cancellation phenomenon, significantly reducing the dispersion of the eigenvalue stochastic response.

Relationship between the stochastic response COV of beams/columns and the material property COV: A positive and approximately linear correlation exists between the fluctuation level (COV) of the input material and the output response COV. This serves as a crucial basis for preliminary reliability prediction problems.

Sensitivity of spatial material randomness and boundary condition types: Sensitivity comparisons indicate that the spatial randomness of the material plays a decisive role over boundary conditions in inducing critical load dispersion. This highlights the importance of material quality control in stability problems.

CONCLUSIONS AND RECOMMENDATIONS

I. Novel Contributions of the Dissertation

This dissertation researched and developed the Stochastic Finite Element Method (SFEM) to analyze the stochastic mechanical responses of beam and column structures with spatially varying random material properties. The main scientific contributions of the dissertation are summarized as follows:

1. **Developing and validating a stochastic finite element analysis framework for beam and column structures with multi-dimensional spatially random material properties:**

- ✓ Integration of multi-dimensional random fields: Incorporating stationary multi-dimensional (2D, 3D) Gaussian random fields into the stochastic mechanical models of beams and columns, thereby broadening the analytical scope compared to traditional one-dimensional (1D) random models;
- ✓ Establishment of analytical tools: Developing the weighted integral technique coupled with the first-order Taylor perturbation expansion to formulate the stochastic stiffness and mass matrices. This approach effectively solves stochastic static, stochastic linear dynamic, and stochastic Euler buckling problems. Furthermore, it directly determines the lower-order statistical moments (expectation, variance, coefficient of variation) of the structural responses without relying on computationally expensive iterative methods;
- ✓ Verification of reliability: The robustness of the proposed SFEM model was rigorously validated by assessing mesh convergence, verifying the asymptotic convergence to deterministic solutions, and cross-validating the results with Monte Carlo Simulations (MCS) and existing literature.

2. Elucidating the stochastic physical mechanisms governing structural responses through correlation lengths and spatial dimensions (2D, 3D):

- ✓ Quantification of the statistical averaging effect: The thesis quantitatively evaluated the influence of spatial correlation lengths on the dispersion of stochastic mechanical responses. At small correlation lengths, local fluctuations are neutralized by the statistical averaging effect, maintaining the coefficient of variation (COV) at a low level. Conversely, as the correlation length increases, the dispersion is amplified and ultimately saturates when the spatial random field degenerates into a global random variable;
- ✓ Discovery of directional dominance: It is demonstrated that the random variation along the longitudinal axis exerts the most dominant influence on the overall dispersion of structural responses, significantly outweighing the effects of variations along the cross-sectional directions.

3. Evaluating the influence of cross-correlation between the random fields of elastic modulus and mass density on the stochastic linear dynamic characteristics of beams:

- ✓ Analysis of statistical interaction mechanisms: Clarifying the statistical interaction between the stiffness matrix and the mass matrix under the influence of the cross-correlation coefficient between the elastic modulus and mass density random fields;
- ✓ Identification of dispersion amplification: Confirming that the COV of eigenvalues (natural frequencies) is amplified and reaches its maximum when the two material property random fields are perfectly negatively correlated;
- ✓ Determination of uncertainty reduction mechanisms: Proving that the dispersion of eigenvalues significantly decreases when the two material property random fields are perfectly positively correlated, providing a crucial quantitative basis for the stochastic linear dynamic analysis of beam structures.

4. Establishing predictive rules and confirming the dominant role of spatial variability in the stochastic analysis of beams and columns:

- ✓ Establishment of predictive rules: Quantitative analysis reveals a positively correlated, approximately linear relationship between the coefficient of variation (COV) of the input material parameters and the COV of the output responses, providing a foundation for the rapid prediction of system uncertainty;
- ✓ Confirmation of the dominant role: Through sensitivity analysis in the Euler buckling problem, the research proved that the spatial variability of material properties dictates the dispersion of the critical buckling load, exhibiting a profoundly greater impact than that of boundary conditions.

II. Limitations of the Dissertation

- ✓ The SFEM model developed in this dissertation is fundamentally based on the Euler–Bernoulli beam theory.
- ✓ The local effects of the spatial random fields have not been deeply considered, and the variation in the neutral axis position during the analysis process has been neglected. Scenarios involving highly fluctuating (strongly heterogeneous) materials have not yet been examined.

III. Recommendations for Future Research

To expand the applicability and robustness of the proposed method, the following research directions are recommended:

- ✓ Investigating the variation of the neutral axis position during the stochastic analysis process;
- ✓ Developing the SFEM based on the Higher-order Shear Deformation Theory (HSDT) to analyze deep beams;
- ✓ Expanding the computational model to incorporate other stochastic factors, such as applied loads, geometric dimensions, boundary conditions, or time-dependent material property degradation;
- ✓ Integrating Deep Learning models with the SFEM to predict and evaluate the stochastic responses of complex structures.

LIST OF SCIENTIFIC PUBLICATIONS

PUBLICATIONS IN ENGLISH

1. **Nguyen Dang Diem**, Ta Duy Hien (2025), "Stochastic finite element for stability of columns considering the two-dimensional random field of elastic modulus", *Results in Engineering* 25, pp. 104496. (Indexed in WoS/ESCI, Q1).
2. **Nguyen Dang Diem**, Ta Duy Hien (2025), "Free vibration of beams using stochastic finite element method considering three-dimensional randomness of material properties", *Forces in Mechanics* 19, pp. 100312. (Indexed in WoS/ESCI, Q2).
3. **Nguyen Dang Diem**, Pham Van Dat, Ta Duy Hien (2025), "Analysis of Beams with a Three-dimensional Random Field of the Modulus of Elasticity Using the Stochastic Finite Element Method", *International Journal of Mathematical, Engineering and Management Sciences* 10(5), pp. 1518-1538. (Indexed in WoS/ESCI, Q2).
4. **Nguyen Dang Diem**, Ta Duy Hien (2025), "Monte Carlo Simulation of a Beam Resting on an Elastic Foundation Considering the Two-Dimensional Stochastic Properties of the Elastic Modulus", *International Journal of Integrated Engineering* 17(5), pp. 65-76. (Indexed in WoS/ESCI, Q3).
5. **Nguyen Dang Diem**, Ta Duy Hien, Dao Sy Dan, Nguyen Duy Hung (2025), "Column Buckling under Stochastic Three-dimensional Material Properties: A Stochastic Finite Element Analysis", *Coupled Systems Mechanics* 14(4), pp. 347-369. (Indexed in WoS/ESCI, Q4).
6. **Nguyen Dang Diem**, Dao Sy Dan, Ta Duy Hien, Dao Ngoc Tien (2025), "Stochastic Stability Analysis of Columns with Randomly Elastic Joint Ends and Two-Dimensional Random Material Properties Using Monte Carlo Simulation", *Journal of Materials Engineering Structures* 12(1), pp. 73-83. (Indexed in WoS/ESCI).
7. **Nguyen Dang Diem**, Ta Duy Hien, Nguyen Trong Hiep, Dao Ngoc Tien (2025), "Analysis of free vibrations of a continuous beam considering the random stiffness of the supports and the two-dimensional random material properties using monte carlo simulation", *Journal of Mechanics of Continua and Mathematical Sciences* 20(4), pp. 1-19. (Indexed in Scopus, Q4).

SCIENTIFIC RESEARCH PROJECTS

1. Principal Investigator, University-level Research Project: "*Stability analysis of columns with three-dimensional randomly varying elastic modulus*", Project No. T2025-PHIL_CT-006, University of Transport and Communications, 2025.